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On the Center Conditions of Certain Fifth Systems

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

There are two basic problems in the qualitative theory of planar nonlinear differential equations, namely the center focus and the limit cycle. The decision problem of the center focus is also the premise and foundation of the study on the limit cycle. Therefore, the study of the center focus and the limit cycle constitutes an independent branch of mathematics.
So far, many methods have been tried to settle the problems of the center focus of the polynomial system. However, the center focus problem of high degree polynomial systems has not been completely solved. In this paper, we use the Poincaré method to study the center focus problem of the five periodic differential equation. Then we use the Alwash-Lloyd method [1,2,3] to derive the center conditions for this differential system.

Keywords: Centers; periodic solutions; composition condition.

LEMMA [2]. Consider the Abel differential equation [4]

$$\frac{dx}{dt} = A(t)x^2 + B(t)x^3, \tag{1}$$

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Where, $A(t+\omega)=A(t)$ and $B(t+\omega)=B(t)$, (ω is a positive constant). The origin is a center for the two-dimensional system if and only if all solutions of the Abel equation starting near the origin are periodic with period 2π. In this case, we say that $x=0$ is a center for the Abel equation. The origin is a center when the coefficients satisfy the following condition

$$\begin{aligned} A(t) &= u'(t)A_1(u(t)), \\ B(t) &= u'(t)B_1(u(t)), \end{aligned} \tag{2}$$

Where $u(t)$ is a periodic function of period 2π, A_1, B_1 are continuous functions. This condition is called the composition condition [5-8].

Consider the fifth polynomial system

$$\begin{cases} \dot{x} = -y + x(P_1(x,y) + P_4(x,y)), \\ \dot{y} = x + y(P_1(x,y) + P_4(x,y)), \end{cases} \tag{3}$$

with $P_n(x,y)=\sum_{i+j=n} P_{ij}x^i y^j$, P_{ij} are real constants. In this paper, we give a short proof to the following result of [1,9].

$$3P_{40} + P_{22} + 3P_{04} = 0 \tag{4}$$

$$(P_{10}{}^2 - P_{01}{}^2)(P_{04} - P_{40}) - P_{10}P_{01}(P_{13} + P_{31}) = 0 \tag{5}$$

$$P_{10}{}^4 P_{04} + P_{01}{}^4 P_{40} - P_{10}P_{01}(P_{10}{}^2 P_{13} + P_{01}{}^2 P_{31} - P_{10}P_{01}P_{22}) = 0 \tag{6}$$

Proof. The system (3) in polar coordinates r and θ becomes

$$\begin{cases} \dot{r} = r^2 P_1(\cos\theta, \sin\theta) + r^5 P_4(\cos\theta, \sin\theta), \\ \dot{\theta} = 1, \end{cases}$$

With,

$$\begin{aligned} P_1(\cos\theta,\sin\theta) &= P_{10}\cos\theta + P_{01}\sin\theta, \\ P_4(\cos\theta,\sin\theta) &= P_{40}\cos^4\theta + P_{31}\cos^3\theta\sin\theta + P_{22}\cos^2\theta\sin^2\theta + P_{13}\cos\theta\sin^3\theta + P_{04}\sin^4\theta. \end{aligned}$$

The origin is a center for (3) if and only if the polynomial differential equation

$$\frac{dr}{d\theta} = r^2 P_1(\cos\theta,\sin\theta) + r^5 P_4(\cos\theta,\sin\theta), \tag{7}$$

have 2π - periodic solution in a neighborhood of $r=0$.

Let $r(\theta, c)$ be solution of (7) with $r(0,c) = c(0 \prec |c| \prec\prec 1)$ .We write

$$r(\theta,c)=\sum_{n=1}^{\infty}a_n(\theta)c^n \quad , \tag{8}$$

Where, $a_1(0)=1$ and $a_n(0)=0, n\geq 2.$

The origin is a center for (5) if and only if $a_1(2\pi)=1, a_n(2\pi)=0, n\geq 2, n\in Z^+$.

Substituting (8) into (7)

$$\dot{a}_1c+\dot{a}_2c^2+\cdots+\dot{a}_nc^n+\cdots=P_1(a_1c+a_2c^2+\cdots+a_nc^n+\cdots)^2+P_4(a_1c+a_2c^2+\cdots+a_nc^n+\cdots)^5$$

Equating the coefficients of c yield

$$\dot{a}_n=P_1\sum_{i+j=n}a_ia_j+P_4\sum_{i+j+k+l+m=n}a_ia_ja_ka_la_m, a_n(0)=0, n\geq 2, n\in Z^+. \tag{9}$$

Solving (9) gives

$$\begin{aligned}
&a_1=1, a_2=\overline{P}_1, a_3=\overline{P}_1^2, a_4=\overline{P}_1^3, a_5=\overline{P}_1^4+\overline{P}_4, a_6=\overline{P}_1^5+2\overline{P}_1\overline{P}_4+3\overline{\overline{P}_1P_4},\\
&a_7=\overline{P}_1^6+3\overline{P}_1^2\overline{P}_4+6\overline{P}_1\overline{\overline{P}_1P_4}+6\overline{\overline{P}_1^2P_4},\\
&a_8=\overline{P}_1^7+4\overline{P}_1^3\overline{P}_4+9\overline{P}_1^2\overline{\overline{P}_1P_4}+12\overline{P}_1\overline{\overline{P}_1^2P_4}+10\overline{\overline{P}_1^3P_4},
\end{aligned}$$

And

$$a_9=\overline{P}_1^8+5\overline{P}_1^4\overline{P}_4+12\overline{P}_1^3\overline{\overline{P}_1P_4}+18\overline{P}_1^2\overline{\overline{P}_1^2P_4}+20\overline{P}_1\overline{\overline{P}_1^3P_4}+\frac{5}{2}\overline{P}_4^{\,2}+15\overline{\overline{P}_1^4P_4}.$$

We know $a_2(2\pi)=a_3(2\pi)=a_4(2\pi)=a_6(2\pi)=a_8(2\pi)=0.$

A bar over a function denotes its indefinite integral.

The three necessary conditions for a center are $a_5(2\pi)=0, a_7(2\pi)=0$ and $a_9(2\pi)=0.$

Be equivalent to $\int_0^{2\pi}P_4d\theta=0, \int_0^{2\pi}\overline{P}_1^2P_4d\theta=0, \int_0^{2\pi}\overline{P}_1^4P_4d\theta=0.$

We have

condition (I): $3P_{40}+P_{22}+3P_{04}=0,$

condition (II): $(P_{10}{}^2-P_{01}{}^2)(P_{04}-P_{40})-P_{10}P_{01}(P_{13}+P_{31})=0,$

and

condition (III): $P_{10}{}^4P_{04}+P_{01}{}^4P_{40}-P_{10}P_{01}(P_{10}{}^2P_{13}+P_{01}{}^2P_{31}-P_{10}P_{01}P_{22})=0.$

We prove that three conditions are also sufficient. Suppose these conditions hold: $P_4 = P_1(\lambda_1 \overline{P_1} + \lambda_3 \overline{P_1}^3)$

$$P_4 = P_1\Big(\frac{P_{40}}{P_{10}}\cos^3\theta + \frac{P_{10}P_{31} - P_{01}P_{40}}{P_{10}^{\ 2}}\cos^2\theta\sin\theta + \frac{P_{10}^{\ 2}P_{22} - P_{10}P_{01}P_{31} + P_{01}^{\ 2}P_{40}}{P_{10}^{\ 3}}\cos\theta\sin^2\theta + \frac{P_{04}}{P_{01}}\sin^3\theta\Big),$$

If and only if

$$\begin{cases} 3P_{10}^{\ 2}P_{04} + P_{10}P_{01}P_{31} - P_{01}^{\ 2}P_{40} = 4\lambda_1 P_{10}^{\ 3}P_{01} + 3\lambda_3 P_{10}^{\ 5}P_{01} + 3\lambda_3 P_{10}^{\ 3}P_{01}^{\ 3} & (10) \\ P_{10}^{\ 2}P_{40} - P_{10}^{\ 2}P_{22} + P_{10}P_{01}P_{31} - P_{01}^{\ 2}P_{40} = 3\lambda_3 P_{10}^{\ 5}P_{01} - \lambda_3 P_{10}^{\ 3}P_{01}^{\ 3} & (11) \\ -P_{10}^{\ 2}P_{04} - P_{01}^{\ 2}P_{40} + P_{10}P_{01}P_{31} = -\lambda_3 P_{10}^{\ 5}P_{01} + 3\lambda_3 P_{10}^{\ 3}P_{01}^{\ 3} & (12) \end{cases}$$

Solving (10) and (11) gives $\lambda_1 = \dfrac{-3P_{10}^{\ 4}P_{40} + P_{10}^{\ 2}P_{01}^{\ 2}P_{22} - P_{10}P_{01}^{\ 3}P_{31} + P_{01}^{\ 4}P_{40}}{3P_{10}^{\ 5}P_{01} - P_{10}^{\ 3}P_{01}^{\ 3}}$,

$$\lambda_3 = \frac{P_{10}^{\ 2}P_{40} - P_{10}^{\ 2}P_{22} + P_{10}P_{01}P_{31} - P_{01}^{\ 2}P_{40}}{3P_{10}^{\ 5}P_{01} - P_{10}^{\ 3}P_{01}^{\ 3}}.$$

Substituting λ_1 and λ_3 into (12), we have

$$P_{10}^{\ 4}P_{40} + P_{10}^{\ 3}P_{01}P_{31} - 2P_{10}^{\ 2}P_{01}^{\ 2}P_{04} - 4P_{10}^{\ 2}P_{01}^{\ 2}P_{40} - P_{10}P_{01}^{\ 3}P_{31} + P_{01}^{\ 4}P_{40} = 0$$

It is known from condition (Ⅱ) and condition (Ⅲ) that the above equation is constant .

Competing Interests

Author has declared that no competing interests exist.

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