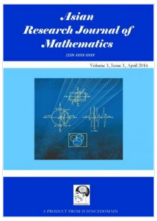

Asian Research Journal of Mathematics

10(4): 1-17, 2018; Article no.ARJOM.43408

ISSN: 2456-477X

The Convolution Sums with the Number of Representations of a Positive Integer as Sum of 6 Squares

Aeran Kim[1*]

[1] *A Private Mathematics Academy, 1-#406, 23, Maebong 5-gil, Deokjin-gu, Jeonju-si, Jeollabuk-do, 54921, Republic of Korea.*

Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2018/43408

Editor(s):

(1) Dr. Junjie Chen, Department of Electrical Engineering, University of Texas at Arlington, USA.

Reviewers:

(1) Maria Isabel Garcia-Planas, Polytechnic University of Catalonia, Spain.

(2) Airault Helene, University of Picardie Jules Verne, France.

(3) Leonardo Simal Moreira, University Centre of Volta Redonda, Brazil.

Complete Peer review History: http://www.sciencedomain.org/review-history/26394

Received: 19 June 2018

Accepted: 29 August 2018

Published: 27 September 2018

Original Research Article

Abstract

For $a_1, a_2, \cdots, a_6 \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we define

$$R_6(a_1, a_2, \cdots, a_6; n) = \text{card}\{(x_1, x_2, \cdots, x_6) \in \mathbb{Z}^6 | n = a_1x_1^2 + a_2x_2^2 + \cdots + a_6x_6^2\}$$

and mainly we obtain some convolution sums

$$\sum_{m=1}^{n-1} R_6(1,1,1,1,4,4;4m)R_6(1,1,1,1,4,4;4n-4m),$$

$$\sum_{m=1}^{n} R_6(1,1,1,4,4,4;4m-2)R_6(1,1,1,4,4,4;4n-4m+2),$$

$$\sum_{m=1}^{2n-1} R_6(1,1,1,1,1,4;2m)R_6(1,1,1,1,1,4;4n-2m),$$

and etc.

**Corresponding author: E-mail: ae_ran_kim@hotmail.com*

Keywords: Divisor functions; infinite product sums; convolution sums.

2010 Mathematics Subject Classification: 11E25, 11F27.

1 Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ denote the sets of positive integers, integers, and complex numbers respectively. For $n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{C}$ with $|q| < 1$, we define

$$\sigma_k(n) = \sum_{d|n} d^k \qquad \text{and} \qquad A(q) := \sum_{n=1}^{\infty} a(n)q^n = q\prod_{n=1}^{\infty}(1-q^{2n})^{12}.$$

Then we can clearly know that

$$a(n) = 0 \qquad \text{for even } n. \tag{1.1}$$

We set for $a_1, a_2, \cdots, a_k \in \mathbb{N}$, $k \in \mathbb{N}$, and $n \in \mathbb{N} \cup \{0\}$

$$R_k(a_1, a_2, \cdots, a_k; n) := \text{card}\{(x_1, x_2, \cdots, x_k) \in \mathbb{Z}^k | n = a_1x_1^2 + a_2x_2^2 + \cdots + a_kx_k^2\}. \tag{1.2}$$

Briefly we write (1.2) as

$$R_k(1, 1, \cdots, 1; n) := R_k(n).$$

For example, Jacobi's classical results [1, §§40–42, p. 159–170] are that

$$R_4(n) = 8\sigma_1(n) - 32\sigma_1(\frac{n}{4}) \qquad \text{and} \qquad R_8(n) = 16\sigma_3(n) - 32\sigma_3(\frac{n}{2}) + 256\sigma_3(\frac{n}{4}).$$

Moreover Glaisher [2, p. 480], [3] showed a formula for $R_{12}(n)$ in 1907. A simple proof of a formula equivalent to Glaisher's formula is given in Williams [4] by

$$R_{12}(n) = 8\sigma_5(n) - 512\sigma_5(\frac{n}{4}) + 16a(n). \tag{1.3}$$

Next we require

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \qquad \text{and} \qquad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

where $q \in \mathbb{C}$ such that $|q| < 1$. Related to (1.2) we can see that

$$\sum_{n=0}^{\infty} R_k(a_1, a_2, \cdots, a_k; n)q^n = \varphi(q^{a_1})\varphi(q^{a_2})\cdots\varphi(q^{a_k})$$

for instance, it is obvious by (1.3)

$$\varphi^{12}(q) = 1 + \sum_{n=1}^{\infty} R_{12}(n)q^n = 1 + \sum_{n=1}^{\infty}\Big(8\sigma_5(n) - 512\sigma_5(\frac{n}{4}) + 16a(n)\Big)q^n. \tag{1.4}$$

In this article we pay attention to the Legendre-Jacobi-Kronecker symbol for discriminant -4, that is

$$\left(\frac{-4}{n}\right) := \begin{cases} 0, & \text{if } n \equiv 0 \pmod 2, \\ 1, & \text{if } n \equiv 1 \pmod 4, \\ -1, & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

for $n \in \mathbb{N}$ and dealing with $\left(\frac{-4}{n}\right)$ we construct

$$G_4(n) := \sum_{d|n} \left(\frac{-4}{n/d}\right) d^2 \qquad \text{and} \qquad H_4(n) := \sum_{d|n} \left(\frac{-4}{d}\right) d^2$$

as Alaca et al. [5, (1.3)]. Using $G_4(n)$ and $H_4(n)$ we deduce as follows :

Lemma 1.1. *Let $n \in \mathbb{N}$. Then*

(a)

$$\sum_{m=1}^{n} H_4(2m-1)G_4(2n-2m+1) = a(n),$$

(b)

$$\sum_{m=1}^{n} H_4(2m-1)H_4(2n-2m+1) = \frac{1}{2}\sigma_5(n) - \frac{65}{2}\sigma_5(\frac{n}{2}) + 32\sigma_5(\frac{n}{4}) + \frac{1}{2}a(n),$$

(c)

$$\sum_{m=1}^{n} G_4(2m-1)G_4(2n-2m+1) = \frac{1}{2}\sigma_5(n) - \frac{1}{2}\sigma_5(\frac{n}{2}) + \frac{1}{2}a(n).$$

Now it is definite that

$$G_4(2n) = 4G_4(n) \qquad \text{and} \qquad H_4(2n) = H_4(n). \tag{1.5}$$

In addition to $G_4(n)$ and $H_4(n)$, we need the function

$$I(n) := \sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ n=x^2+4y^2}} \left(x^2 - 4y^2\right), \qquad n \equiv 1 \pmod 2. \tag{1.6}$$

And we note that $x^2 + 4y^2 \equiv 0$ or $1 \pmod 4$ for $(x, y) \in \mathbb{Z}^2$ so that

$$I(n) = 0, \qquad \text{if } n \equiv 3 \pmod 4. \tag{1.7}$$

We obtain Theorem 1.2 as one of applications of (1.2) :

Theorem 1.2. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\sum_{m=1}^{2n-1} R_6(1,1,1,1,1,4;2m)R_6(1,1,1,1,1,4;4n-2m)$$
$$= 1088\sigma_5(n) - 1080\sigma_5(\frac{n}{2}) - 512\sigma_5(\frac{n}{4}) - 192G_4(n) + 8H_4(n) + 696a(n),$$

(b)

$$\sum_{m=1}^{2n-1} R_6(1,1,1,1,4,4;2m)R_6(1,1,1,1,4,4;4n-2m)$$
$$= 320\sigma_5(n) - 312\sigma_5(\frac{n}{2}) - 512\sigma_5(\frac{n}{4}) - 64G_4(n) + 8H_4(n) + 312a(n),$$

(c)

$$\sum_{m=1}^{2n-1} R_6(1,1,1,4,4,4;2m)R_6(1,1,1,4,4,4;4n-2m)$$
$$= 80\sigma_5(n) - 72\sigma_5(\frac{n}{2}) - 512\sigma_5(\frac{n}{4}) - 32G_4(n) + 8H_4(n) + 88a(n),$$

(d)

$$\sum_{m=1}^{2n-1} R_6(1,1,4,4,4,4;2m)R_6(1,1,4,4,4,4;4n-2m)$$
$$= 16\sigma_5(n) - 8\sigma_5(\frac{n}{2}) - 512\sigma_5(\frac{n}{4}) - 32G_4(n) + 8H_4(n) + 24a(n),$$

(e)

$$\sum_{m=1}^{2n-1} R_6(1,4,4,4,4,4;2m)R_6(1,4,4,4,4,4;4n-2m)$$
$$= 8\sigma_5(n) - 512\sigma_5(\frac{n}{4}) - 32G_4(n) + 8H_4(n) + 16a(n),$$

(f)

$$\sum_{m=1}^{2n-1} R_6(1,1,1,2,2,4;2m)R_6(1,1,1,2,2,4;4n-2m)$$
$$= 256\sigma_5(n) - 248\sigma_5(\frac{n}{2}) - 512\sigma_5(\frac{n}{4}) - 128G_4(n) + 8H_4(n) + 120a(n),$$

(g)

$$\sum_{m=1}^{2n-1} R_6(1,1,2,2,4,4;2m)R_6(1,1,2,2,4,4;4n-2m)$$
$$= 64\sigma_5(n) - 56\sigma_5(\frac{n}{2}) - 512\sigma_5(\frac{n}{4}) - 64G_4(n) + 8H_4(n) + 56a(n),$$

(h)

$$\sum_{m=1}^{2n-1} R_6(1,2,2,4,4,4;2m)R_6(1,2,2,4,4,4;4n-2m)$$
$$= 16\sigma_5(n) - 8\sigma_5(\frac{n}{2}) - 512\sigma_5(\frac{n}{4}) - 32G_4(n) + 8H_4(n) + 24a(n).$$

In Section 3 we show :

Theorem 1.3. *Let $q \in \mathbb{C}$ and $n \in \mathbb{N}$. Then we obtain*

(a)

$$q\varphi^8(q)\varphi^2(q^2)\psi^2(q^4) = \sum_{n=1}^{\infty}\left(\frac{1}{2}\sigma_5(n) - \frac{1}{2}\sigma_5(\frac{n}{2}) + \frac{1}{2}a(n)\right)q^n,$$

(b)

$$q\varphi^8(-q)\varphi^2(q^2)\psi^2(q^4) = \sum_{n=1}^{\infty}\left(\frac{1}{2}\sigma_5(n) - \frac{65}{2}\sigma_5(\frac{n}{2}) + 32\sigma_5(\frac{n}{4}) + \frac{1}{2}a(n)\right)q^n.$$

2 Proofs of Lemma 1.1, Theorem 1.2 and Comparable Convolution Sums

The basic properties of $\varphi(q)$ are :

Proposition 2.1. *(See Berndt [6, Eq. (1.3.32), Eq. (3.6.7)]) Let $q \in \mathbb{Q}$ with $|q| < 1$. Then we have*

(a)

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2),$$

(b)

$$\varphi(-q)\varphi(q) = \varphi^2(-q^2).$$

Moreover in Alaca et al. [5, Theorem 2.1] we can see that

$$\sum_{\substack{n=1 \\ n\equiv 1 \ (\mathrm{mod}\ 4)}}^{\infty} I(n)q^n = \frac{1}{8}\varphi^5(q)\varphi(-q) - \frac{1}{8}\varphi(q)\varphi^5(-q). \tag{2.1}$$

Lemma 2.1. *Let $n \in \mathbb{N}$. Then we have*

$$\begin{aligned}
&\sum_{m<\frac{n+1}{2}} I(2m-1)I(n-2m+1) \\
&= -\frac{7}{60}\left((-1)^{\frac{n}{2}}+1\right)\sigma_5(\frac{n}{2}) + \frac{1}{30}\left(80(-1)^{\frac{n}{2}}+151\right)\sigma_5(\frac{n}{4}) \\
&\quad + \frac{16}{15}\left(15(-1)^{\frac{n}{2}}-22\right)\sigma_5(\frac{n}{8}) + \frac{1}{4}\left((-1)^n+1\right)\sigma_3(n) - \frac{9}{2}\sigma_3(\frac{n}{2}) \\
&\quad + \frac{2}{3}\left\{-3\left((-1)^{\frac{n}{2}}-1\right)n + 2\left((-1)^{\frac{n}{2}}+2\right)\right\}\sigma_3(\frac{n}{4}) \\
&\quad - \left((-1)^{\frac{n}{2}}-1\right)\sigma_1(\frac{n}{8}) - \frac{1}{2}\left((-1)^{\frac{n}{2}}-7\right)a(\frac{n}{2}).
\end{aligned}$$

Proof. In advance we have shown that

$$\begin{aligned}
&\varphi^2(q)\varphi^{10}(-q)+\varphi^{10}(q)\varphi^2(-q)\\
&=2+\sum_{n=1}^{\infty}\Big\{\frac{16}{15}\left(8(-1)^{\frac{n}{2}}-7\right)\sigma_5(\frac{n}{2})+\frac{32}{15}\left(80(-1)^{\frac{n}{2}}+151\right)\sigma_5(\frac{n}{4})\\
&\quad-\frac{22528}{15}\sigma_5(\frac{n}{8})+16\left((-1)^n+1\right)\sigma_3(n)-288\sigma_3(\frac{n}{2})\\
&\quad-\frac{128}{3}\left\{3\left((-1)^{\frac{n}{2}}-1\right)n-2\left((-1)^{\frac{n}{2}}+2\right)\right\}\sigma_3(\frac{n}{4})\\
&\quad-64\left((-1)^{\frac{n}{2}}-1\right)\sigma_1(\frac{n}{8})+224a(\frac{n}{2})\Big\}q^n
\end{aligned} \tag{2.2}$$

with all $n \in \mathbb{N}$ in Kim [7, (26)] also by (1.4) and Proposition 2.1 (b), we obtain

$$\begin{aligned}
\varphi^6(q)\varphi^6(-q)&=\varphi^{12}(-q^2)\\
&=1+\sum_{n=1}^{\infty}\left(8\sigma_5(n)-512\sigma_5(\frac{n}{4})+16a(n)\right)(-q^2)^n\\
&=1+\sum_{n=1}^{\infty}\left(8(-1)^{\frac{n}{2}}\sigma_5(\frac{n}{2})-512(-1)^{\frac{n}{2}}\sigma_5(\frac{n}{8})+16(-1)^{\frac{n}{2}}a(\frac{n}{2})\right)q^n.
\end{aligned} \tag{2.3}$$

Therefore by (1.6), (1.7), and (2.1) we can show that

$$\begin{aligned}
&\sum_{N=1}^{\infty}\left(\sum_{m=1}^{N}I(2m-1)I(2N-2m+1)\right)q^{2N}\\
&=\left(\sum_{m=1}^{\infty}I(2m-1)q^{2m-1}\right)\left(\sum_{n=1}^{\infty}I(2n-1)q^{2n-1}\right)\\
&=\left(\sum_{\substack{m=1\\ m\equiv 1 \pmod 2}}^{\infty}I(m)q^m\right)\left(\sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty}I(n)q^n\right)\\
&=\left(\sum_{\substack{m=1\\ m\equiv 1 \pmod 4}}^{\infty}I(m)q^m\right)\left(\sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty}I(n)q^n\right)\\
&=\left(\frac{1}{8}\varphi^5(q)\varphi(-q)-\frac{1}{8}\varphi(q)\varphi^5(-q)\right)^2\\
&=\frac{1}{64}\left(\varphi^{10}(q)\varphi^2(-q)+\varphi^2(q)\varphi^{10}(-q)-2\varphi^6(q)\varphi^6(-q)\right)
\end{aligned}$$

and so we apply (2.2) and (2.3).

□

Proposition 2.2. *(See Kim [7, Theorem 1.2]) Let $n \in \mathbb{N}$. Then we have*

(a)

$$\sum_{m=1}^{n-1}H_4(m)G_4(n-m)=\frac{1}{4}G_4(n)-\frac{1}{4}a(n),$$

(b)

$$\sum_{m=1}^{n-1} H_4(m)H_4(n-m) = \frac{1}{2}\sigma_5(\frac{n}{2}) - 32\sigma_5(\frac{n}{4}) + \frac{1}{2}H_4(n) - \frac{1}{2}a(n),$$

(c)

$$\sum_{m=1}^{n-1} G_4(m)G_4(n-m) = \frac{1}{32}\sigma_5(n) - \frac{1}{32}\sigma_5(\frac{n}{2}) - \frac{1}{32}a(n).$$

Proposition 2.2 is essential to deduce some various convolution sums in whole parts. First of all there is a relation of divisor functions as

$$\sigma_k(pn) - \left(p^k + 1\right)\sigma_k(n) + p^k\sigma_k(\frac{n}{p}) = 0 \tag{2.4}$$

for a prime p and $k, n \in \mathbb{N}$ in Williams [8, Theorem 3.1(ii)].

Proposition 2.3. *(See Alaca et al. [5, Theorem 1.1]) Let $n \in \mathbb{N}$. Then we have*

(a)

$$R_6(1,1,1,1,1,4;n) = \begin{cases} 6G_4(n) + 2I(n), & \text{if } n \equiv 1 \pmod 4, \\ 10G_4(n), & \text{if } n \equiv 2,3 \pmod 4, \\ 6G_4(n) - 4H_4(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

(b)

$$R_6(1,1,1,1,4,4;n) = \begin{cases} 4G_4(n) + 2I(n), & \text{if } n \equiv 1 \pmod 4, \\ 4G_4(n), & \text{if } n \equiv 3 \pmod 4, \\ 6G_4(n), & \text{if } n \equiv 2 \pmod 4, \\ 2G_4(n) - 4H_4(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

(c)

$$R_6(1,1,1,4,4,4;n) = \begin{cases} 3G_4(n) + \frac{3}{2}I(n), & \text{if } n \equiv 1 \pmod 4, \\ G_4(n), & \text{if } n \equiv 3 \pmod 4, \\ 3G_4(n), & \text{if } n \equiv 2 \pmod 4, \\ G_4(n) - 4H_4(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

(d)

$$R_6(1,1,4,4,4,4;n) = \begin{cases} 2G_4(n) + I(n), & \text{if } n \equiv 1 \pmod 4, \\ 0, & \text{if } n \equiv 3 \pmod 4, \\ G_4(n), & \text{if } n \equiv 2 \pmod 4, \\ G_4(n) - 4H_4(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

(e)

$$R_6(1,4,4,4,4,4;n) = \begin{cases} G_4(n) + \frac{1}{2}I(n), & \text{if } n \equiv 1 \pmod 4, \\ 0, & \text{if } n \equiv 2,3 \pmod 4, \\ G_4(n) - 4H_4(n), & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

(f)

$$R_6(1,1,1,2,2,4;n) = \begin{cases} 4G_4(n) + I(n), & \textit{if } n \equiv 1 \pmod 4, \\ 4G_4(n), & \textit{if } n \equiv 2,3 \pmod 4, \\ 4G_4(n) - 4H_4(n), & \textit{if } n \equiv 0 \pmod 4. \end{cases}$$

(g)

$$R_6(1,1,2,2,4,4;n) = \begin{cases} 2G_4(n) + I(n), & \textit{if } n \equiv 1 \pmod 4, \\ 2G_4(n), & \textit{if } n \equiv 2,3 \pmod 4, \\ 2G_4(n) - 4H_4(n), & \textit{if } n \equiv 0 \pmod 4. \end{cases}$$

(h)

$$R_6(1,2,2,4,4,4;n) = \begin{cases} G_4(n) + \frac{1}{2}I(n), & \textit{if } n \equiv 1 \pmod 4, \\ G_4(n), & \textit{if } n \equiv 2,3 \pmod 4, \\ G_4(n) - 4H_4(n), & \textit{if } n \equiv 0 \pmod 4. \end{cases}$$

Proof of Lemma 1.1. (a) By (1.1), (1.5), and Proposition 2.2 (a) let us expand

$$\begin{aligned} &\sum_{m=1}^{2n-1} H_4(m)G_4(2n-m) \\ &= \sum_{m=1}^{n-1} H_4(2m)G_4(2n-2m) + \sum_{m=1}^{n} H_4(2m-1)G_4(2n-2m+1) \\ &= 4\sum_{m=1}^{n-1} H_4(m)G_4(n-m) + \sum_{m=1}^{n} H_4(2m-1)G_4(2n-2m+1) \end{aligned}$$

and so

$$\begin{aligned} &\sum_{m=1}^{n} H_4(2m-1)G_4(2n-2m+1) \\ &= \sum_{m=1}^{2n-1} H_4(m)G_4(2n-m) - 4\sum_{m=1}^{n-1} H_4(m)G_4(n-m) \\ &= a(n) - \frac{1}{4}a(2n) \\ &= a(n). \end{aligned}$$

(b) In a similar manner to Lemma 1.1 (a), we can know that

$$\begin{aligned} &\sum_{m=1}^{2n-1} H_4(m)H_4(2n-m) \\ &= \sum_{m=1}^{n-1} H_4(2m)H_4(2n-2m) + \sum_{m=1}^{n} H_4(2m-1)H_4(2n-2m+1) \\ &= \sum_{m=1}^{n-1} H_4(m)H_4(n-m) + \sum_{m=1}^{n} H_4(2m-1)H_4(2n-2m+1) \end{aligned}$$

and so appealing to (1.1), (1.5), and Proposition 2.2 (b) we have

$$\begin{aligned}
&\sum_{m=1}^{n} H_4(2m-1)H_4(2n-2m+1)\\
&= \sum_{m=1}^{2n-1} H_4(m)H_4(2n-m) - \sum_{m=1}^{n-1} H_4(m)H_4(n-m)\\
&= \frac{1}{2}\sigma_5(n) - \frac{65}{2}\sigma_5(\frac{n}{2}) + 32\sigma_5(\frac{n}{4}) + \frac{1}{2}H_4(2n) - \frac{1}{2}H_4(n) - \frac{1}{2}a(2n) + \frac{1}{2}a(n)\\
&= \frac{1}{2}\sigma_5(n) - \frac{65}{2}\sigma_5(\frac{n}{2}) + 32\sigma_5(\frac{n}{4}) + \frac{1}{2}a(n).
\end{aligned}$$

(c) Now from (1.1), (1.5), Proposition 2.2 (c), and (2.4) we can deduce that

$$\begin{aligned}
&\sum_{m=1}^{n} G_4(2m-1)G_4(2n-2m+1)\\
&= \sum_{m=1}^{2n-1} G_4(m)G_4(2n-m) - 16\sum_{m=1}^{n-1} G_4(m)G_4(n-m)\\
&= \frac{1}{32}\sigma_5(2n) - \frac{17}{32}\sigma_5(n) + \frac{1}{2}\sigma_5(\frac{n}{2}) - \frac{1}{32}a(2n) + \frac{1}{2}a(n)\\
&= \frac{1}{32}\left\{33\sigma_5(n) - 32\sigma_5(\frac{n}{2})\right\} - \frac{17}{32}\sigma_5(n) + \frac{1}{2}\sigma_5(\frac{n}{2}) + \frac{1}{2}a(n)\\
&= \frac{1}{2}\sigma_5(n) - \frac{1}{2}\sigma_5(\frac{n}{2}) + \frac{1}{2}a(n).
\end{aligned}$$

□

Theorem 2.2. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\begin{aligned}
&\sum_{m=1}^{n-1} R_6(1,1,1,1,1,4;4m)R_6(1,1,1,1,1,4;4n-4m)\\
&= 288\sigma_5(n) - 280\sigma_5(\frac{n}{2}) - 512\sigma_5(\frac{n}{4}) - 192G_4(n) + 8H_4(n) - 104a(n),
\end{aligned}$$

(b)

$$\begin{aligned}
&\sum_{m=1}^{n-1} R_6(1,1,1,1,4,4;4m)R_6(1,1,1,1,4,4;4n-4m)\\
&= 32\sigma_5(n) - 24\sigma_5(\frac{n}{2}) - 512\sigma_5(\frac{n}{4}) - 64G_4(n) + 8H_4(n) + 24a(n),
\end{aligned}$$

(c)

$$\begin{aligned}
&\sum_{m=1}^{n-1} R_6(1,1,1,4,4,4;4m)R_6(1,1,1,4,4,4;4n-4m)\\
&= 8\sigma_5(n) - 512\sigma_5(\frac{n}{4}) - 32G_4(n) + 8H_4(n) + 16a(n),
\end{aligned}$$

(d)

$$\sum_{m=1}^{n-1} R_6(1,1,4,4,4,4;4m)R_6(1,1,4,4,4,4;4n-4m)$$
$$= 8\sigma_5(n) - 512\sigma_5(\frac{n}{4}) - 32G_4(n) + 8H_4(n) + 16a(n),$$

(e)

$$\sum_{m=1}^{n-1} R_6(1,4,4,4,4,4;4m)R_6(1,4,4,4,4,4;4n-4m)$$
$$= 8\sigma_5(n) - 512\sigma_5(\frac{n}{4}) - 32G_4(n) + 8H_4(n) + 16a(n),$$

(f)

$$\sum_{m=1}^{n-1} R_6(1,1,1,2,2,4;4m)R_6(1,1,1,2,2,4;4n-4m)$$
$$= 128\sigma_5(n) - 120\sigma_5(\frac{n}{2}) - 512\sigma_5(\frac{n}{4}) - 128G_4(n) + 8H_4(n) - 8a(n),$$

(g)

$$\sum_{m=1}^{n-1} R_6(1,1,2,2,4,4;4m)R_6(1,1,2,2,4,4;4n-4m)$$
$$= 32\sigma_5(n) - 24\sigma_5(\frac{n}{2}) - 512\sigma_5(\frac{n}{4}) - 64G_4(n) + 8H_4(n) + 24a(n),$$

(h)

$$\sum_{m=1}^{n-1} R_6(1,2,2,4,4,4;4m)R_6(1,2,2,4,4,4;4n-4m)$$
$$= 8\sigma_5(n) - 512\sigma_5(\frac{n}{4}) - 32G_4(n) + 8H_4(n) + 16a(n).$$

Proof. Since the proofs are similar therefore we only prove (a). From (1.5) and Proposition 2.3 (a) we note that

$$\sum_{m=1}^{n-1} R_6(1,1,1,1,1,4;4m)R_6(1,1,1,1,1,4;4n-4m)$$
$$= \sum_{m=1}^{n-1} \Big\{6G_4(4m) - 4H_4(4m)\Big\}\Big\{6G_4(4(n-m)) - 4H_4(4(n-m))\Big\}$$
$$= \sum_{m=1}^{n-1} \Big\{6\cdot 16G_4(m) - 4H_4(m)\Big\}\Big\{6\cdot 16G_4(n-m) - 4H_4(n-m)\Big\}$$
$$= (6\cdot 16)^2 \sum_{m=1}^{n-1} G_4(m)G_4(n-m) - 6\cdot 16\cdot 4\sum_{m=1}^{n-1} G_4(m)H_4(n-m)$$
$$- 4\cdot 6\cdot 16\sum_{m=1}^{n-1} H_4(m)G_4(n-m) + 4\cdot 4\sum_{m=1}^{n-1} H_4(m)H_4(n-m)$$

and so we appeal to Proposition 2.2.

□

Theorem 2.3. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\sum_{m=1}^{n} R_6(1,1,1,1,1,4;4m-2)R_6(1,1,1,1,1,4;4n-4m+2)$$
$$= 800\sigma_5(n) - 800\sigma_5(\frac{n}{2}) + 800a(n),$$

(b)

$$\sum_{m=1}^{n} R_6(1,1,1,1,4,4;4m-2)R_6(1,1,1,1,4,4;4n-4m+2)$$
$$= 288\sigma_5(n) - 288\sigma_5(\frac{n}{2}) + 288a(n),$$

(c)

$$\sum_{m=1}^{n} R_6(1,1,1,4,4,4;4m-2)R_6(1,1,1,4,4,4;4n-4m+2)$$
$$= 72\sigma_5(n) - 72\sigma_5(\frac{n}{2}) + 72a(n),$$

(d)

$$\sum_{m=1}^{n} R_6(1,1,4,4,4,4;4m-2)R_6(1,1,4,4,4,4;4n-4m+2)$$
$$= 8\sigma_5(n) - 8\sigma_5(\frac{n}{2}) + 8a(n),$$

(e)

$$\sum_{m=1}^{n} R_6(1,4,4,4,4,4;4m-2)R_6(1,4,4,4,4,4;4n-4m+2) = 0,$$

(f)

$$\sum_{m=1}^{n} R_6(1,1,1,2,2,4;4m-2)R_6(1,1,1,2,2,4;4n-4m+2)$$
$$= 128\sigma_5(n) - 128\sigma_5(\frac{n}{2}) + 128a(n),$$

(g)

$$\sum_{m=1}^{n} R_6(1,1,2,2,4,4;4m-2)R_6(1,1,2,2,4,4;4n-4m+2)$$
$$= 32\sigma_5(n) - 32\sigma_5(\frac{n}{2}) + 32a(n),$$

(h)

$$\sum_{m=1}^{n} R_6(1,2,2,4,4,4;4m-2)R_6(1,2,2,4,4,4;4n-4m+2)$$
$$= 8\sigma_5(n) - 8\sigma_5(\frac{n}{2}) + 8a(n).$$

Proof. Because the proofs are similar thus we only prove (a). From (1.5) and Proposition 2.3 (a) we observe that

$$\begin{aligned}
&\sum_{m=1}^{n} R_6(1,1,1,1,1,4;4m-2)R_6(1,1,1,1,1,4;4n-4m+2)\\
&=\sum_{m=1}^{n} 10G_4(4m-2)\cdot 10G_4(4n-4m+2)\\
&=(10\cdot 4)^2\sum_{m=1}^{n} G_4(2m-1)G_4(2n-2m+1)
\end{aligned}$$

and so we apply Lemma 1.1 (c).

□

Proof of Theorem 1.2. Proofs are similar so that we only prove (a). We note that

$$\begin{aligned}
&\sum_{m=1}^{2n-1} R_6(1,1,1,1,1,4;2m)R_6(1,1,1,1,1,4;4n-2m)\\
&=\sum_{m=1}^{n-1} R_6(1,1,1,1,1,4;4m)R_6(1,1,1,1,1,4;4n-4m)\\
&\quad+\sum_{m=1}^{n} R_6(1,1,1,1,1,4;4m-2)R_6(1,1,1,1,1,4;4n-4m+2)
\end{aligned}$$

and so we use Theorem 2.2 (a) and Theorem 2.3 (a).

□

3 Proof of Theorem 1.3 and Other Results

Corollary 3.1. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\sum_{m=1}^{n} G_4(4m-1)G_4(4n-4m+1) = 8\sigma_5(n) - 8\sigma_5(\frac{n}{2}),$$

(b)

$$\sum_{m=1}^{n} H_4(4m-1)H_4(4n-4m+1) = -8\sigma_5(n) + 8\sigma_5(\frac{n}{2}).$$

Proof. Proofs are similar so we only prove (a). Let us replace n with $2n$ in Lemma 1.1 (c) as

$$\begin{aligned}
&\sum_{m=1}^{2n} G_4(2m-1)G_4(4n-2m+1)\\
&=\sum_{m=1}^{n} G_4(4m-1)G_4(4n-4m+1)+\sum_{m=1}^{n} G_4(4m-3)G_4(4n-4m+3)\\
&=2\sum_{m=1}^{n} G_4(4m-1)G_4(4n-4m+1)
\end{aligned}$$

and so

$$\sum_{m=1}^{n} G_4(4m-1)G_4(4n-4m+1) = \frac{1}{2}\sum_{m=1}^{2n} G_4(2m-1)G_4(4n-2m+1).$$

□

Proposition 3.1. *(See [5, Theorem 2.6]) For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(a)

$$\sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty} G_4(n)q^n = \frac{1}{2}q\left(\varphi^4(q^2)+\varphi^4(-q^2)\right)\varphi(q^4)\psi(q^8),$$

(b)

$$\sum_{\substack{n=1 \\ n\equiv 3 \pmod 4}}^{\infty} G_4(n)q^n = \frac{1}{2}q\left(\varphi^4(q^2)-\varphi^4(-q^2)\right)\varphi(q^4)\psi(q^8).$$

Proof of Theorem 1.3. First by Corollary 3.1 (a) and Proposition 3.1 we have

$$\begin{aligned}
&\sum_{N=1}^{\infty}\left(\sum_{m=1}^{N} G_4(4m-1)G_4(4N-4m+1)\right)q^{4N} \\
&= \sum_{N=1}^{\infty}\left(8\sigma_5(N)-8\sigma_5(\frac{N}{2})\right)q^{4N} \\
&= \left(\sum_{m=1}^{\infty} G_4(4m-1)q^{4m-1}\right)\left(\sum_{n=0}^{\infty} G_4(4n+1)q^{4n+1}\right) \\
&= \left(\sum_{\substack{m=1 \\ m\equiv 3 \pmod 4}}^{\infty} G_4(m)q^m\right)\left(\sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty} G_4(n)q^n\right) \\
&= \frac{1}{2}q\left(\varphi^4(q^2)-\varphi^4(-q^2)\right)\varphi(q^4)\psi(q^8)\cdot\frac{1}{2}q\left(\varphi^4(q^2)+\varphi^4(-q^2)\right)\varphi(q^4)\psi(q^8) \\
&= \frac{1}{4}q^2\left(\varphi^8(q^2)-\varphi^8(-q^2)\right)\varphi^2(q^4)\psi^2(q^8) \\
&= \frac{1}{4}q^2\varphi^8(q^2)\varphi^2(q^4)\psi^2(q^8)-\frac{1}{4}q^2\varphi^8(-q^2)\varphi^2(q^4)\psi^2(q^8)
\end{aligned}$$

and so

$$\begin{aligned}
&\sum_{N=1}^{\infty}\left(8\sigma_5(N)-8\sigma_5(\frac{N}{2})\right)q^{4N} \\
&= \frac{1}{4}q^2\varphi^8(q^2)\varphi^2(q^4)\psi^2(q^8)-\frac{1}{4}q^2\varphi^8(-q^2)\varphi^2(q^4)\psi^2(q^8).
\end{aligned}$$

Reducing q^2 to q in the above identity, we deduce that

$$q\varphi^8(q)\varphi^2(q^2)\psi^2(q^4) - q\varphi^8(-q)\varphi^2(q^2)\psi^2(q^4) = \sum_{n=1}^{\infty}\left(32\sigma_5(n) - 32\sigma_5(\frac{n}{2})\right)q^{2n}$$
$$= \sum_{n=1}^{\infty}\left(32\sigma_5(\frac{n}{2}) - 32\sigma_5(\frac{n}{4})\right)q^{n}. \tag{3.1}$$

Second from Lemma 1.1 (c) and Proposition 3.1 we obtain

$$\sum_{N=1}^{\infty}\left(\sum_{m=1}^{2N-1} G_4(2m-1)G_4(2(2N-1)-2m+1)\right)q^{4N-2}$$
$$= \sum_{N=1}^{\infty}\left(\frac{1}{2}\sigma_5(2N-1) - \frac{1}{2}\sigma_5(\frac{2N-1}{2}) + \frac{1}{2}a(2N-1)\right)q^{4N-2}$$
$$= \sum_{N=1}^{\infty}\left(\sum_{m=1}^{N-1} G_4(4m-1)G_4(4N-4m-1)\right)q^{4N-2}$$
$$+ \sum_{N=1}^{\infty}\left(\sum_{m=1}^{N} G_4(4m-3)G_4(4N-4m+1)\right)q^{4N-2}$$
$$= \left(\sum_{m=1}^{\infty} G_4(4m-1)q^{4m-1}\right)\left(\sum_{n=1}^{\infty} G_4(4n-1)q^{4n-1}\right)$$
$$+ \left(\sum_{m=1}^{\infty} G_4(4m-3)q^{4m-3}\right)\left(\sum_{n=0}^{\infty} G_4(4n+1)q^{4n+1}\right)$$
$$= \left(\sum_{\substack{m=1\\ m\equiv 3 (\mathrm{mod} 4)}}^{\infty} G_4(m)q^{m}\right)\left(\sum_{\substack{n=1\\ n\equiv 3 (\mathrm{mod} 4)}}^{\infty} G_4(n)q^{n}\right)$$
$$+ \left(\sum_{\substack{m=1\\ m\equiv 1 (\mathrm{mod} 4)}}^{\infty} G_4(m)q^{m}\right)\left(\sum_{\substack{n=1\\ n\equiv 1 (\mathrm{mod} 4)}}^{\infty} G_4(n)q^{n}\right)$$
$$= \left\{\frac{1}{2}q\left(\varphi^4(q^2) - \varphi^4(-q^2)\right)\varphi(q^4)\psi(q^8)\right\}^2 + \left\{\frac{1}{2}q\left(\varphi^4(q^2) + \varphi^4(-q^2)\right)\varphi(q^4)\psi(q^8)\right\}^2$$
$$= \frac{1}{2}q^2\varphi^8(q^2)\varphi^2(q^4)\psi^2(q^8) + \frac{1}{2}q^2\varphi^8(-q^2)\varphi^2(q^4)\psi^2(q^8)$$

and so

$$\sum_{N=1}^{\infty}\left(\frac{1}{2}\sigma_5(2N-1) - \frac{1}{2}\sigma_5(\frac{2N-1}{2}) + \frac{1}{2}a(2N-1)\right)q^{4N-2}$$
$$= \frac{1}{2}q^2\varphi^8(q^2)\varphi^2(q^4)\psi^2(q^8) + \frac{1}{2}q^2\varphi^8(-q^2)\varphi^2(q^4)\psi^2(q^8).$$

Applying $\sigma_k(\frac{\text{odd}}{2}) = 0$ and lowering q^2 into q in the above equation, moreover using (1.1) and (2.4) we have

$$
\begin{aligned}
& q\varphi^8(q)\varphi^2(q^2)\psi^2(q^4) + q\varphi^8(-q)\varphi^2(q^2)\psi^2(q^4) \\
&= \sum_{n=1}^{\infty} \left(\sigma_5(2n-1) + a(2n-1)\right) q^{2n-1} \\
&= \sum_{\substack{n=1 \\ 2\nmid n}}^{\infty} \sigma_5(n)q^n + \sum_{\substack{n=1 \\ 2\nmid n}}^{\infty} a(n)q^n \\
&= \sum_{n=1}^{\infty} \sigma_5(n)q^n - \sum_{\substack{n=1 \\ 2|n}}^{\infty} \sigma_5(n)q^n + \sum_{n=1}^{\infty} a(n)q^n - \sum_{\substack{n=1 \\ 2|n}}^{\infty} a(n)q^n \\
&= \sum_{n=1}^{\infty} \sigma_5(n)q^n - \sum_{n=1}^{\infty} \sigma_5(2n)q^{2n} + \sum_{n=1}^{\infty} a(n)q^n \\
&= \sum_{n=1}^{\infty} \sigma_5(n)q^n - \sum_{n=1}^{\infty} \Big(33\sigma_5(n) - 32\sigma_5(\frac{n}{2})\Big) q^{2n} + \sum_{n=1}^{\infty} a(n)q^n \\
&= \sum_{n=1}^{\infty} \Big(\sigma_5(n) - 33\sigma_5(\frac{n}{2}) + 32\sigma_5(\frac{n}{4}) + a(n)\Big) q^n.
\end{aligned} \tag{3.2}
$$

Finally adding and subtracting (3.1) and (3.2) we prove (a) and (b).

□

4 Conclusion

In this paper we consider the number of representations of a positive integer as sum of 6 squares. After this, we construct some convolution sums with those representations and deduce their formulas which contain divisor functions and the Legendre-Jacobi-Kronecker symbol and the infinite product sums.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Jacobi CGJ. Fundamenta nova theoriae functionum ellipticarum. Bornträger, Regiomonti,. Gesammelte Werke, Erster Band, Chelsea Publishing Co. New York. 1969;49-239.

[2] Glaisher JWL. On the numbers of representations of a number as a sum of $2r$ squares, where $2r$ does not exceed eighteen. Proc. London Math. Soc. 1907;5:479-490.

[3] Glaisher JWL. On the representations of a number as the sum of two, four, six, eight, ten and twelve squares. Quart. J. Pure and Appl. Math. 1907;38:1-62.

[4] Williams KS. On Liouville's twelve squares theorem. Far East J. Math. Sci. 2008;29:239-242.

[5] Alaca A, Alaca S, Williams KS. Sextenary quadratic forms and an identity of Klein and Fricke. International Journal of Number Theory. 2010;6(1):169-183.

[6] Berndt BC. Number Theory in the Spirit of Ramanujan. American Mathematical Society (AMS), Providence; 2006.

[7] Kim A. Evaluation of certain convolution sums involving the Legendre-Jacobi-Kronecker symbol for discriminant -4. Asian Journal of Mathematics and Computer Research. 2015;5(1):38-59.

[8] Williams KS. Number Theory in the Spirit of Liouville. London Mathematical Society, Student Texts 76, Cambridge; 2011.

APPENDIX

The first twenty values of $a(n)$ are listed in the following table.

Table 1. $a(n)$ **for** n **(**$1 \le n \le 20$**)**

n	$a(n)$	n	$a(n)$	n	$a(n)$	n	$a(n)$
1	1	6	0	11	540	16	0
2	0	7	-88	12	0	17	594
3	-12	8	0	13	-418	18	0
4	0	9	-99	14	0	19	836
5	54	10	0	15	-648	20	0

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sciencedomain.org/review-history/26394