British Journal of Applied Science & Technology
2(2): 112-131, 2012

SCIENCEDOMAIN *international*
www.sciencedomain.org

An Inventory Model for Deteriorating Items with Quadratic Demand and Partial Backlogging

R. Begum[1*], S. K. Sahu[2] and R. R. Sahoo[3]

[1]Department of Mathematics, Padmanaa College of Engg., Rourkela-769002, Odisha, India.
[2]Department of Statistics, Sambalpur University, Jyotivihar, Burla, Sambalpur-768048, India.
[3]Department of Physics, Synergy Institute of Technology, Bhubaneswar-752101, India.

Research Article

Received 11th February 2012
Accepted 5th April 2012
Online Ready 19th April 2012

ABSTRACT

Aims: The scope of the model lies in its applicability in the management inventories of time-quadratic demand. It is also seen that large pile of goods displayed in a supermarket will motivate the customer to buy more. So the presence of inventory has a motivational effect on the people around it. Also there may be occasional shortages in inventory due to many reasons. Therefore, we develop an EOQ model for the inventory of a deteriorating item, taking demand rate and allowing shortages in inventory.
Study Design: This paper presents an inventory model for deteriorating items with quadratic demand. In the model, shortages are allowed and partially backordered. The backlogging rate is a variable and dependent on the waiting time for the next replenishment. A numerical example is taken to illustrate the model and the sensitivity analysis is also studied.
Methodology: Our purpose is to devise a mathematical model on inventory management taking all these factors into consideration.
Results: Convexity condition of the cost function is established to ensure the existence of unique point of minimum.
Conclusion: The proposed model can be extended in several ways. For instance, we may extend the demand function to stochastic fluctuating demand patterns or stock-dependent demand rate. Finally, we could extend the model to incorporate some more realistic features such as quantity discounts, permissible delay in payments, time value of money and inflation etc.

Keywords: Inventory; quadratic demand; deteriorating items; partial backlogging.

**Corresponding author: Email: begum_rehena@rediffmail.com;*

1. INTRODUCTION

The effect of deterioration of physical goods cannot be disregarded in many inventory systems. In daily life, the deterioration of goods is a common phenomenon. Deterioration is defined as decay, spoilage, damage, evaporation, obsolescence, pilferage, loss of utility or marginal value of a commodity that results in decreasing usefulness from the original one. For example pharmaceuticals, chemicals, foodstuff etc. deteriorate significantly. The deterioration rate of inventory in stock during the storage period constitutes an important factor, which has attracted the researchers. Whitin (1957) is the first author who studied an inventory model for fashion goods deteriorating at the end of a prescribed storage period. An exponentially decaying inventory was developed by Ghare and Schrader (1963). Emmons (1968) established a replenishment model for radioactive nuclide generators. Shah and Jaiswal (1977) established an order-level-inventory model for perishable items with a constant rate of deterioration. The deterioration occurs as soon as the retailer receives the commodities that have assumed in all inventory models for deteriorating items. But in real life situation, most of goods would have a span of maintaining quality or the original condition. During that period, there was no occurrence of deterioration. These items are fish, fruit, meat, vegetables, alcohols, blood and gasoline. This phenomenon is termed as “non-instantaneous deterioration”. Jeyaraman and Sugapriya (2008) developed an ELSP for non-instantaneous deteriorating items using price discount.

In practice, some customers would like to wait for backlogging during the shortage period, but the others would not. Consequently, the opportunity cost due to lost sales should be considered in the modeling. Many researchers (Murdeshwar, 1988; Goyal et al., 1992; Hariga, 1996; Chakrabarti and Chaudhuri, 1997; Hariga and Alyan, 1997) assumed that shortages are completely backlogged. Chang and Dye (1999) argued that the backlogging rate should be dependent on the length of the waiting time. They were the first to give a definition for the time-dependent partial backlogging rate. Wee (1995) developed an article in the field of deteriorating items with shortages has revealed the economic order quantity with a known market demand rate. Researchers like Sana (2010a) and Roy et al. (2011a, 2011b, 2011c) considered the case of partial backlogging rates in their models. However, in some inventory system, for many stocks such as fashionable commodities, the length of the waiting time for the next replenishment becomes main factor for determining whether the backlogging will be accepted or not. The longer the waiting time, the smaller would be the backlogging rate. Therefore, the backlogging rate is variable and is dependent on the waiting time for the next replenishment. In this paper, the backlogging rate was assumed to be a fixed fraction of demand rate during the shortage period.

Goyal and Giri (2001) provided a detailed review of deteriorating inventory literatures. They indicated: the assumption of constant demand rate is not always applicable to many inventory items like fashionable clothes, electronic goods etc. As they experience fluctuations in the demand rate. Many products experience a period of rising demand during the growth phase of their product life cycle. Time-varying demands were first considered by Silver and Meal (1969). In this model, they established the Heuristic solution procedure. Many research articles by Ritchie (1980, 1984, 1985); Deb and Chaudhuri (1986); Dave (1989a, 1989b, 1989c); Hariga (1993); Chung and Ting (1993); Kim (1995); Jalan and Chaudhuri (1999b); Lin et al. (2000) etc., analyzed linear time-varying demand. Aggarwal and Bahari-Kashani (1991); Hollier and Mak (1983); Hariga and Benkherouf (1994); Wee (1995); Jalan and Chaudhuri (1999a) etc., developed inventory models in which exponential time-varying demand has been taken. With the progress of time, researchers developed inventory models with time-dependent demand rates. Later, Ghosh and Chaudhuri (2004);

Khanra and Chaudhuri (2003) etc., established their models in which quadratic time-varying demand was considered. Sana (2010b) considered the case of ameliorating items and deteriorating items in a multi-item EOQ model where the demand is being influenced by enterprises. Recently Begum et al. (2010) has developed an EOQ model with quadratic demand with Weibull distribution deterioration.

During the first three decades, many marketing researchers observed that in some retailer systems like supermarket, the demand of goods might be influenced by the on-hand inventory. For example, Levin et al. (1972) pointed out that at times, the presence of inventory has a motivational effect on the people around it. It is a common belief that a large pile of goods displayed in a supermarket will lead the customer to buy more. Dave and Patel (1981) developed an inventory model for deteriorating items with time-proportional demand. This model was extended by Sachan (1984) to cover the backlogging option. Donaldson (1977) discussed the classical no-shortage inventory policy for linear, time-dependent demand for the first time. Sana (2011a, 2011b, 2011c) considered the case of price sensitive demands in his models. In the opinion of many authors, an alternative or perhaps more realistic approach is to consider quadratic time-dependence of demand. This demand may represent all types of time-dependence depending on the signs of the parameters of the time-quadratic demand function. Khanra et al. (2010) considered an EOQ model with stock and price dependent demand rate. Sana and Chaudhuri (2004) developed production policy for a deteriorating item with time-dependent demand and shortages. The present work attempts to model the situation where the demand rate is a continuous function of time and items deteriorate at a constant rate. Here shortages are allowed and the backlogging rate is variable and dependent on the waiting time for the next replenishment. In the present paper, we have extended the work of Sahu et al. (2007) by taking the demand rate to be quadratic function of time. The purpose of the present paper is to give a new approach to the inventory literature or time-dependent demand patterns. Quadratic function of time is the best representation of accelerated growth (or decline) in the demand. An analytical solution of the model is discussed and it is illustrated with the help of a numerical example. Sensitivity of the optimal solution with respect to change in different parameter values is also carried out. The model ends with a suitable conclusion.

2. ASSUMPTIONS AND NOTATIONS

The assumptions are as follows:

1. The lead time is zero.
2. Replenishment rate is infinite.
3. The demand rate $D(t)$ at any time $'t'$ is given by

$$D(t) = \begin{cases} a + bt + ct^2, & i(t) > 0 \\ \sim\ , & i(t) \leq 0 \end{cases}$$

   where a, b, c $\left(a \geq 0, b \neq 0, c \neq 0\right)$ and $\sim$ are positive constants and $i(t)$ is the inventory level at time t.
4. Shortages are allowed. For convenient, a fraction of demand is backlogged. If the waiting time longer, then the backlogging rate will be smaller. Let's assume $B(t)$ be the fraction where t is the waiting time up to the next replenishment. We

consider $B(t) = \frac{1}{1+¢\, t}$, where ¢ is known as the backlogging parameter which is a positive constant.

5. t_d is the length of time in which the product has no deterioration. (i. e. fresh product time). A constant fraction ‾ $(0 <$ ‾ $< 1)$ of the on-hand inventory deteriorates after this period and there is no replacement of the deteriorated units. t_d and ‾ are given constants.
6. t_1 is the length of time in which the inventory has no shortages. The length of order cycle and the order quantity per cycle is given by T and Q respectively. Thus the decision variables are t_1, T and Q.
7. $i_1(t)$ is the inventory level at time t $(0 \le t \le t_d)$ in which the product has no deterioration. $i_2(t)$ is the inventory level at time t $(t_d \le t \le t_1)$ in which the product has deterioration. $i_3(t)$ is the inventory level at time t $(t_1 \le t \le T)$ in which the product has shortage.

Notations

- A is the constant ordering cost per order
- c_1 is the constant inventory holding cost per unit time
- c_2 is the constant deteriorating cost per unit
- c_3 is the constant shortage cost for backlogged items
- c_4 is the constant lost sale cost per unit
- $TVC(t_1, T)$ is the total relevant inventory cost per unit time of inventory system.
- t_1^*, T^*, Q^* and TVC^* is the optimal values of length of time, length of order cycle, order quantity and total relevant inventory cost.

3. MATHEMATICAL DEVELOPMENT

This paper is developed by the consideration of the replenishment problem of a single non-instantaneous deteriorating item with partial backlogging. The inventory model runs as follows:

Let $i_{\max}$ be the units of item arrive at the inventory system at the beginning of each cycle. The inventory level decreases to quadratic demand rate during the time interval $[0, t_d]$. The inventory level decreases both due to demand and deterioration till it becomes zero in the interval $[t_d, t_1]$. The shortage interval keeps to the end of the current order cycle. The total process is repeated. The inventory level at different instants of time is shown in figure 1.

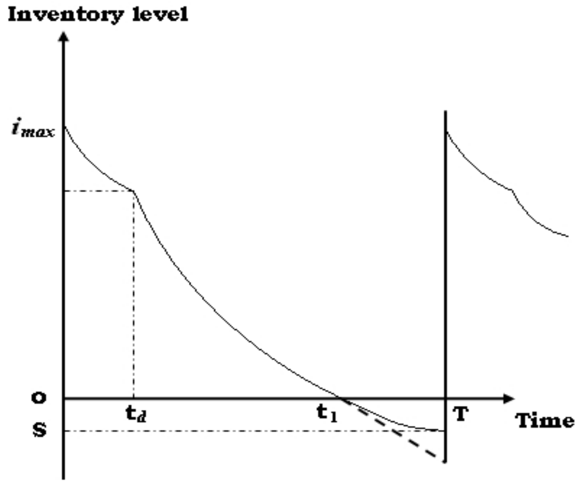

Fig. 1. Graphical presentation of inventory system

As described above, the inventory level decreases owing to quadratic demand rate during the time interval $[0, t_d]$. The inventory level $i_1(t)$ is governed by the following differential equation:

$$\frac{di_1(t)}{dt} = -\left(a + bt + ct^2\right),\ 0 \le t \le t_d \tag{3.1}$$

having the boundary condition $i_1(0) = i_{\max}$. The solution of equation (3.1) is:

$$i_1(t) = i_{\max} - \left(at + b\frac{t^2}{2} + c\frac{t^3}{3}\right),\ 0 \le t \le t_d \tag{3.2}$$

The inventory level decreases not only due to the quadratic demand rate but also due to the deterioration during the interval $[t_d, t_1]$. The inventory level is governed by the differential equation:

$$\frac{di_2(t)}{dt} +^{-} i_2(t) = -\left(a + bt + ct^2\right),\ t_d \le t \le t_1 \tag{3.3}$$

having the boundary condition $i_2(t_1) = 0$. Soling equation(3. 3) for $i_2(t)$, which yields

$$i_2(t) = e^{-(t_1 - t)}\left[\frac{a}{-} + b\left(\frac{t_1}{-} - \frac{1}{-^{2}}\right) + c\left(\frac{t_1^{\,2}}{-} - \frac{2t_1}{-^{2}} + \frac{2}{-^{3}}\right)\right] - \left[\frac{a}{-} + b\left(\frac{t}{-} - \frac{1}{-^{2}}\right) + c\left(\frac{t^2}{-} - \frac{2t}{-^{2}} + \frac{2}{-^{3}}\right)\right]$$

$$t_d \le t \le t_1 \tag{3.4}$$

Considering continuity of $i(t)$ at $t = t_d$, it follows from equation (3. 2) and (3. 4) that

$$i_1(t_d) = i_2(t_d) = i_{\max} - \left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right)$$

$$= e^{-(t_1 - t_d)}\left[\frac{a}{-} + b\left(\frac{t_1}{-} - \frac{1}{-^{2}}\right) + c\left(\frac{t_1^{\,2}}{-} - \frac{2t_1}{-^{2}} + \frac{2}{-^{3}}\right)\right] - \left[\frac{a}{-} + b\left(\frac{t_d}{-} - \frac{1}{-^{2}}\right) + c\left(\frac{t_d^{\,2}}{-} - \frac{2t_d}{-^{2}} + \frac{2}{-^{3}}\right)\right]$$

The maximum inventory level for each cycle is given by,

$$i_{\max} = \left[\frac{a}{-} + b\left(\frac{t_1}{-} - \frac{1}{-^2} \right) + c\left(\frac{t_1^{\ 2}}{-} - \frac{2t_1}{-^2} + \frac{2}{-^3} \right) \right] e^{-\ (t_1 - t_d)} + \left[at_d + b\frac{t_d^{\ 2}}{2} + c\frac{t_d^{\ 3}}{3} \right]$$
$$- \left[\frac{a}{-} + b\left(\frac{t_d}{-} - \frac{1}{-^2} \right) + c\left(\frac{t_d^{\ 2}}{-} - \frac{2t_d}{-^2} + \frac{2}{-^3} \right) \right] \quad (3.5)$$

Substituting equation (3. 5) in equation (3.2), it becomes

$$i_1(t) = \left[\frac{a}{-} + b\left(\frac{t_1}{-} - \frac{1}{-^2} \right) + c\left(\frac{t_1^{\ 2}}{-} - \frac{2t_1}{-^2} + \frac{2}{-^3} \right) \right] e^{-\ (t_1 - t_d)}$$
$$+ \left[a(t_d - t) + \frac{b}{2}\left(t_d^{\ 2} - t^2\right) + \frac{c}{3}\left(t_d^{\ 3} - t^3\right) \right] - \left[\frac{a}{-} + b\left(\frac{t_d}{-} - \frac{1}{-^2} \right) + c\left(\frac{t_d^{\ 2}}{-} - \frac{2t_d}{-^2} + \frac{2}{-^3} \right) \right]$$

$$0 \le t \le t_d \quad (3.6)$$

During the shortage interval $[t_1, T]$, the demand at time 't' is partially backlogged at fraction $\frac{1}{1+\text{¢}\,(T-t)}$. The differential equation governing the amount of demand backlogged is given by,

$$\frac{di_3(t)}{dt} = -\sim B(T-t) = -\frac{\sim}{1+\text{¢}\,(T-t)}, \ t_1 \le t \le T \quad (3.7)$$

having the boundary condition $i_3(t_1) = 0$. Soling equation (3. 7), which yields

$$i_3(t) = \frac{-\sim}{\text{¢}} \{ \ln[1+\text{¢}\,(T-t_1)] - \ln[1+\text{¢}\ \ (T-t)] \}, \ t_1 \le t \le T \quad (3.8)$$

Putting $t = T$ in equation (3. 8), we obtain the maximum amount of demand backlogged per cycle as,

$$S = -i_3(T) = \frac{\sim}{\text{¢}} \ \ln[1+\text{¢}\,(T-t_1)] \quad (3.9)$$

Let Q be the order quantity per cycle and is obtained from equation (3. 5) and (3. 9) as

$$Q = i_{\max} + S = \left[\frac{a}{-} + b\left(\frac{t_1}{-} - \frac{1}{-^2} \right) + c\left(\frac{t_1^{\ 2}}{-} - \frac{2t_1}{-^2} + \frac{2}{-^3} \right) \right] e^{-\ (t_1 - t_d)} + \left[at_d + b\frac{t_d^{\ 2}}{2} + c\frac{t_d^{\ 3}}{3} \right]$$
$$- \left[\frac{a}{-} + b\left(\frac{t_d}{-} - \frac{1}{-^2} \right) + c\left(\frac{t_d^{\ 2}}{-} - \frac{2t_d}{-^2} + \frac{2}{-^3} \right) \right] + \frac{\sim}{\text{¢}} \ln[1+\text{¢}\,(T-t_1)] \quad (3.10)$$

The total relevant cost per cycle consists of five different costs. The total relevant inventory cost per unit time is the sum of all costs per order cycle. Thus, the total relevant cost per unit time is given by,

$$TVC(t_1,T)=\frac{1}{T}\left\{\begin{array}{l}\text{inventory holding cost per cycle}+\text{the deterioration cost per cycle}\\ +\text{shortage cost per cycle due to backlog}+\text{Ordering cost}\\ +\text{opportunity cost per cycle due to lost sales}\end{array}\right\}$$

$$=\frac{1}{T}[A+HC+DC+SC+OC]$$

$$=\frac{1}{T}\left[A+c_1\left\{\int_0^{t_d} i_1(t)dt+\int_{t_d}^{t_1} i_2(t)dt\right\}+c_2\left\{i_2(t_d)-\int_{t_d}^{t_1} D(t)dt\right\}+c_3\left\{\int_{t_1}^{T}[-i_3(t)]dt\right\}\right.$$
$$\left.+c_4\left\{\int_{t_1}^{T}\sim\ [1-B(T-t)]\,dt\right\}\right]$$

$$=\frac{1}{T}\left[A+c_1\left\{t_d\left[\frac{a}{-}+b\left(\frac{t_1}{-}-\frac{1}{-^2}\right)+c\left(\frac{t_1^2}{-}-\frac{2t_1}{-^2}+\frac{2}{-^3}\right)\right]e^{-(t_1-t_d)}+\left(a\frac{t_d^{\,2}}{2}+b\frac{t_d^{\,3}}{3}+c\frac{t_d^{\,4}}{4}\right)\right.\right.$$
$$-t_d\left[\frac{a}{-}+b\left(\frac{t_d}{-}-\frac{1}{-^2}\right)+c\left(\frac{t_d^2}{-}-\frac{2t_d}{-^2}+\frac{2}{-^3}\right)\right]+\left[\frac{a}{-^2}+\frac{b}{-}\left(\frac{t_1}{-}-\frac{1}{-^2}\right)+\frac{c}{-}\left(\frac{t_1^2}{-}-\frac{2t_1}{-^2}+\frac{2}{-^3}\right)\right]\left(e^{-(t_1-t_d)}-1\right)$$
$$\left.-\left[\frac{a}{-}(t_1-t_d)+b\left(\frac{(t_1^{\,2}-t_d^{\,2})}{2-}-\frac{(t_1-t_d)}{-^2}\right)+c\left(\frac{(t_1^{\,3}-t_d^{\,3})}{3-}-\frac{(t_1^{\,2}-t_d^{\,2})}{-^2}+\frac{2(t_1-t_d)}{-^3}\right)\right]\right\}$$
$$+c_2\left\{\left[\frac{a}{-}+b\left(\frac{t_1}{-}-\frac{1}{-^2}\right)+c\left(\frac{t_1^2}{-}-\frac{2t_1}{-^2}+\frac{2}{-^3}\right)\right]e^{-(t_1-t_d)}-\left[\frac{a}{-}+b\left(\frac{t_d}{-}-\frac{1}{-^2}\right)+c\left(\frac{t_d^2}{-}-\frac{2t_d}{-^2}+\frac{2}{-^3}\right)\right]\right.$$
$$\left.\left.-a(t_1-t_d)-\frac{b}{2}(t_1^{\,2}-t_d^{\,2})-\frac{c}{3}(t_1^{\,3}-t_d^{\,3})\right\}+\left(\frac{c_3}{¢}+c_4\right)\sim\left[(T-t_1)-\frac{\ln\{1+¢(T-t_1)\}}{¢}\right]\right]$$

(3.11)

For convenience,

$$M=\left[\frac{a}{-}+b\left(\frac{t_d}{-}-\frac{1}{-^2}\right)+c\left(\frac{t_d^2}{-}-\frac{2t_d}{-^2}+\frac{2}{-^3}\right)\right]>0,\qquad N=\sim\left(\frac{c_3}{¢}+c_4\right)>0,$$

$$R=Mt_d>0,\ P=\left(\frac{a}{2}t_d^2+\frac{b}{3}t_d^3+\frac{c}{4}t_d^4\right)>0$$

Thus, equation (3.11) yields

$$\begin{aligned}TVC(t_1,T)=\frac{1}{T}\bigg[&A+c_1\bigg\{t_d\bigg[\frac{a}{-}+b\left(\frac{t_1}{-}-\frac{1}{-^{2}}\right)+c\left(\frac{t_1^2}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\right)\bigg]e^{-(t_1-t_d)}+\left(a\frac{t_d^{2}}{2}+b\frac{t_d^{3}}{3}+c\frac{t_d^{4}}{4}\right)\\&-R+P+\bigg[\frac{a}{-^{2}}+\frac{b}{-}\left(\frac{t_1}{-}-\frac{1}{-^{2}}\right)+\frac{c}{-}\left(\frac{t_1^2}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\right)\bigg]\left(e^{-(t_1-t_d)}-1\right)\\&-\bigg[\frac{a}{-}(t_1-t_d)+b\left(\frac{(t_1^{2}-t_d^{2})}{2-}-\frac{(t_1-t_d)}{-^{2}}\right)+c\left(\frac{(t_1^{3}-t_d^{3})}{3-}-\frac{(t_1^{2}-t_d^{2})}{-^{2}}+\frac{2(t_1-t_d)}{-^{3}}\right)\bigg]\bigg]\bigg\}\\&+c_2\bigg\{\bigg[\frac{a}{-}+b\left(\frac{t_1}{-}-\frac{1}{-^{2}}\right)+c\left(\frac{t_1^2}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\right)\bigg]e^{-(t_1-t_d)}-M-a(t_1-t_d)-\frac{b}{2}(t_1^{2}-t_d^{2})-\frac{c}{3}(t_1^{3}-t_d^{3})\bigg\}\\&+N\bigg[(T-t_1)-\frac{\ln\{1+\text{¢}(T-t_1)\}}{\text{¢}}\bigg]\bigg]\end{aligned}$$

(3.12)

The total relevant inventory cost per unit time which is found in equation (3. 12) is a function of the two variables t_1 and T. Therefore, the total relevant cost per unit time will be minimum if

$$\frac{\partial TVC(t_1,T)}{\partial t_1}=0 \text{ and } \frac{\partial TVC(t_1,T)}{\partial T}=0 \tag{3.13}$$

From equation (3. 12) and (3. 13), we get the equation

$$\begin{aligned}&\frac{1}{T}\bigg[c_1\bigg\{t_d\,e^{-(t_1-t_d)}\bigg[\frac{b}{-}+c\left(\frac{2t_1}{-}-\frac{2}{-^{2}}\right)\bigg]+\left(e^{-(t_1-t_d)}-1\right)\bigg[\frac{b}{-^{2}}+\frac{c}{-}\left(\frac{2t_1}{-}-\frac{2}{-^{2}}\right)\bigg]+\\&-.t_d.e^{-(t_1-t_d)}\bigg[\frac{a}{-}+b\left(\frac{t_1}{-}-\frac{1}{-^{2}}\right)+c\left(\frac{t_1^{2}}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\right)\bigg]+-.e^{-(t_1-t_d)}\bigg[\frac{a}{-^{2}}+\frac{b}{-}\left(\frac{t_1}{-}-\frac{1}{-^{2}}\right)+\frac{c}{-}\left(\frac{t_1^{2}}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\right)\bigg]\\&-\bigg[\frac{a}{-}+b\left(\frac{t_1}{-}-\frac{1}{-^{2}}\right)+c\left(\frac{t_1^{2}}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\right)\bigg]\bigg]\bigg\}+c_2\bigg\{-e^{-(t_1-t_d)}\bigg[\frac{a}{-}+b\left(\frac{t_1}{-}-\frac{1}{-^{2}}\right)+c\left(\frac{t_1^{2}}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\right)\bigg]\\&-\left(a+bt_1+ct_1^{2}\right)+e^{-(t_1-t_d)}\bigg[\frac{b}{-}+c\left(\frac{2t_1}{-}-\frac{2}{-^{2}}\right)\bigg]\bigg]\bigg\}+N\left(-1+\frac{1}{[1+\text{¢}(T-t_1)]}\right)\bigg]=0\end{aligned}$$

(3.14)

and

$$\frac{N}{T}\left(1-\frac{1}{1+\text{¢}(T-t_1)}\right)-\frac{1}{T^2}\Bigg[A+c_1\Bigg\{t_d e^{-(t_1-t_d)}\left[\frac{a}{\text{¯}}+b\left(\frac{t_1}{\text{¯}}-\frac{1}{\text{¯}^2}\right)+c\left(\frac{t_1^{2}}{\text{¯}}-\frac{2t_1}{\text{¯}^2}+\frac{2}{\text{¯}^3}\right)\right]$$

$$-\frac{a}{\text{¯}}(t_1-t_d)-b\left(\frac{(t_1^{2}-t_d^{2})}{2\text{¯}}-\frac{(t_1-t_d)}{\text{¯}^2}\right)-c\left(\frac{2(t_1-t_d)}{\text{¯}^3}-\frac{(t_1^{2}-t_d^{2})}{\text{¯}^2}+\frac{(t_1^{3}-t_d^{3})}{3\text{¯}}\right)$$

$$+\left(e^{-(t_1-t_d)}-1\right)\left[\frac{a}{\text{¯}^2}+\frac{b}{\text{¯}}\left(\frac{t_1}{\text{¯}}-\frac{1}{\text{¯}^2}\right)+\frac{c}{\text{¯}}\left(\frac{t_1^{2}}{\text{¯}}-\frac{2t_1}{\text{¯}^2}+\frac{2}{\text{¯}^3}\right)\right]+P-R\Bigg\}$$

$$+c_2\left\{e^{-(t_1-t_d)}\left[\frac{a}{\text{¯}}+b\left(\frac{t_1}{\text{¯}}-\frac{1}{\text{¯}^2}\right)+c\left(\frac{t_1^{2}}{\text{¯}}-\frac{2t_1}{\text{¯}^2}+\frac{2}{\text{¯}^3}\right)\right]-M-a(t_1-t_d)-\frac{b}{2}(t_1^{2}-t_d^{2})-\frac{c}{3}(t_1^{3}-t_d^{3})\right\}$$

$$+N\left((T-t_1)-\frac{Log[1+(T-t_1)\text{¢}]}{\text{¢}}\right)\Bigg]=0$$

(3.15)

4. NUMERICAL EXAMPLE

Let $A=100, a=4, b=6, c=7, c_1=30, c_2=15, c_3=25, c_4=10, \sim=20, \text{¢}=5$, $t_d=0.5$ and $\text{¯}=0.5$, in appropriate units. Solving the non-linear equations (3.14) and (3.15); we obtain the optimum values of t_1^{*} and T^{*} are $t_1^{*}=0.601055$ and $T^{*}=1.05244$. Substituting the values of t_1^{*} and T^{*} in (3.10) and (3.11), the optimal order quantity per cycle $Q^{*}=8.75578$ and the minimum total relevant cost per unit time $TVC^{*}(t_1,T)=486.652$ respectively. It is numerically verified that this solution satisfies the convexity condition for $TVC^{*}(t_1,T)$ which is established in Appendix.

5. SENSITIVITY ANALYSIS

We now study the effects of changes in the value of system parameters $A, a, b, c, \text{¯}, \text{¢}$, $\sim, c_1, c_2, c_3, c_4, t_d$ on the optimal length of inventory t_1^{*}, the optimal length of order cycle T^{*}, the optimal order quantity per cycle Q^{*} and the minimum total relevant cost per unit time TVC^{*} of the Example 1. The sensitivity analysis is performed by changing each of the parameter by $50\%, 25\%, 10\%, -10\%, -25\%$ and -50%, taking one parameter at a time and keeping the remaining parameters unchanged. The results are shown in Table 1.

Table 1. Effect of changes in the parameters of the inventory

Parameter	% Change in the parameter	% Change in			
		t_1^*	T^*	Q^*	TVC^*
A	+50	4. 9473	-0. 91406	0. 9882	11. 3555
	+25	-0. 6957	-9. 02134	-4. 87186	8. 7917
	+10	-2. 1344	-8. 8631	-7. 6020	7. 4285
	-10	1. 3482	5. 9594	4. 6991	-4. 8001
	-25	2. 4799	11. 7944	8. 9736	-9. 8721
	-50	3. 6254	18. 9708	13. 6644	-16. 3800
~	+50	19. 6331	19. 5118	52. 3633	1. 5230
	+25	11. 0326	12. 8976	28. 1713	-0. 9892
	+10	4. 9336	6. 5609	12. 3148	-1. 3796
	-10	-5. 9247	-9. 9366	-14. 2740	4. 2817
	-25	-9. 6902	-7. 5122	-21. 4277	-2. 0228
	-50	-26. 2074	-0. 1719	-38. 6903	-11. 7361
a	+50	-7. 4603	-4. 7081	7. 0669	9. 9892
	+25	-3. 9069	-2. 4219	3. 6413	4. 9248
	+10	-1. 5743	-0. 9891	1. 4813	9. 9517
	-10	1. 6163	1. 0195	-1. 5147	-1. 9350
	-25	-4. 1267	2. 6110	-3. 8486	-4. 7999
	-50	-8. 5349	5. 4606	-7. 9026	-9. 4780
b	+50	-5. 8087	-1. 5468	2. 6126	-8. 6396
	+25	-3. 0630	-0. 7933	1. 3793	-4. 2011
	+10	-1. 2674	-0. 3221	0. 5717	-1. 6402
	-10	1. 3281	0. 3297	-0. 6009	-1. 6068
	-25	3. 4449	0. 8371	-1. 5660	-3. 9313
	-50	-7. 3509	1. 7188	-3. 3786	-7. 5532
c	+50	-3. 8716	-0. 2489	1. 1907	32. 9888
	+25	-2. 0364	-0. 0988	0. 6494	16. 4859
	+10	-0. 8410	-0. 0313	-1. 1982	5. 8549
	-10	0. 8791	0. 1710	-0. 3011	-6. 5940
	-20	2. 2754	0. 0855	-0. 8111	-16. 4920
	-50	4. 8311	-0. 1539	-1. 8837	-33. 0342
c_1	+50	-14. 3311	-5. 9588	-8. 1881	35. 7284
	+25	-7. 9415	-3. 2448	-4. 7175	17. 5299
	+10	-3. 4041	-1. 3815	-2. 0854	6. 9258
	-10	3. 7756	1. 5269	2. 4377	-6. 7970
	-25	10. 3170	4. 1902	7. 0117	-16. 7107
	-50	24. 6974	10. 3388	18. 9499	-32. 1722
c_2	+50	-0. 8320	0. 4237	0. 0376	1. 1786
	+25	-0. 4254	0. 2232	0. 0244	0. 6047
	+10	-0. 1725	0. 0921	0. 0111	0. 2457
	-10	0. 1750	-0. 0959	-0. 0132	-0. 2511
	-25	0. 4450	-0. 2489	-0. 0380	-0. 6376
	-50	0. 9089	-0. 5301	-0. 0951	-1. 3087

Table 1 continues……………….

c_3	+50	7. 7688	9. 8200	9. 9449	-1. 3977
	+25	4. 1831	5. 8587	5. 7185	-1. 2996
	+10	1. 7856	2. 7174	2. 5921	-0. 7654
	-10	-2. 1272	-3. 8748	-3. 5812	1. 5524
	-25	-5. 3311	-10. 0878	-9. 3837	4. 6906
	-50	-7. 7144	-9. 0491	-9. 0693	2. 5171
c_4	+50	-14. 0692	15. 4384	16. 7434	-0. 3028
	+25	-7. 7705	9. 8200	9. 9460	-1. 3973
	+10	-3. 4110	4. 8990	4. 7456	-1. 1776
	-10	-4. 9795	-10. 2842	-9. 4691	5. 0845
	-25	-7. 7144	-10. 0491	-9. 0693	2. 5171
	-50	-14. 0494	-5. 8545	-8. 0533	3. 1655
¢	+50	-1. 1464	-4. 0087	-11. 6710	2. 8759
	+25	-0. 7020	-2. 3051	-6. 6920	1. 5900
	+10	-0. 3219	-1. 0119	-2. 9379	0. 6781
	-10	0. 3941	1. 1620	3. 3826	-0. 7428
	-25	1. 1717	3. 2619	9. 5510	-1. 9944
	-50	3. 3031	8. 1743	24. 4492	-4. 5108
t_d	+50	9. 2339	-6. 0287	-2. 5446	12. 9862
	+25	4. 1793	-3. 8377	-2. 3156	8. 1877
	+10	1. 8607	-0. 7639	-0. 3286	3. 1067
	-10	-1. 9212	0. 2964	0. 06395	-3. 2937
	-25	-4. 8128	0. 1738	0. 05619	-8. 8498
	-50	-9. 4809	-1. 0024	0. 3786	-20. 3757
_	+50	-1. 7371	0. 8190	0. 2067	-41. 0761
	+25	-0. 9082	0. 4522	0. 1328	-29. 1127
	+10	-0. 0379	0. 0199	0. 0067	-1. 8952
	-10	+0. 0379	-0. 0199	-0. 0067	1. 8962
	-25	0. 9900	-0. 5739	-0. 2257	89. 7931
	-50	2. 0558	-1. 3273	-0. 6021	-----

On the basis of the results shown in Table 1, the following observations can be made:

a) Q^* decreases while TVC^* increases with increase in the value of the parameter A. The obtained results show that t_1^*, T^* are moderately sensitive to change in the value of A. But TVC^* is highly sensitive to changes in A.

b) t_1^*, T^*, Q^* and TVC^* increase with increase in the value of the parameter $\sim$.

c) t_1^*, T^* decrease with increase in the value of a, b and c. Moreover, it is seen that with increase in parameter a, b and c, the value of Q^*, TVC^* increases. So t_1^*, T^*, Q^*, TVC^* is insensitive to changes in a, b and c.

d) With increase in value of ¯ ; t_1^*, TVC^* decreases while T^*, Q^* increases. t_1^*, T^* and Q^* are less sensitive to ¯ where as TVC^* is highly sensitive.

e) With increase in the parameter c_1, c_2 ; t_1^*, T^* and Q^* decreases. But, TVC^* increases and decreases with rise in parameter c_1, c_2 .

f) t_1^*, T^* and Q^* increase while T^*, Q^* decrease while TVC^* decreases with increase in the value of the parameter c_3 and c_4.

g) T^*, Q decrease while TVC^* increases with increase in the value of the parameter t_d, ¢ . t_1^* increases and decreases with increase in the value of the parameter t_d, ¢ . TVC^* are more sensitive to changes in the value of t_d, ¢ .

6. CONCLUSION

In this paper, an optimal replenishment policy has been considered for non-instantaneous deteriorating items with quadratic demand. Here shortages are allowed and the backlogging rate is variable. The backlogging rate is dependent on the waiting time for the next replenishment. A complete rate or a constant partial rate was used in many studies to describe the backlogging rate. But it is more realistic to assume the backlogging rate to be time proportional with waiting time of backlogging. We have considered here the backlogging rate as $\frac{1}{1+¢\, t}$, which seems to be better from the exponential backlogging rate.

It is a common belief that, a large pile of goods displayed in a supermarket will motivate the customer to buy more. So the presence of inventory has a motivational effect on the people around it. The quadratic time-dependence of demand of the form $D(t) = a + bt + ct^2, a \geq 0, b \neq 0, c \neq 0$ .This type of demand has a better representation of time-varying market. If we compare the other two types of time-dependents like linear and exponential, it is seen that linear time-dependence demand leads to uniform change in the real market. At the same instant, exponential time-dependence demand also seems to be unrealistic because an exponential rate of change is very high and it is in doubt that the market demand of any product may undergo a high rate of change like exponential function. Thus the alternative and probably more realistic approach is to consider the quadratic time-dependence of demand which may represent all types of time-dependence depending on the signs of the parameters of the time-quadratic demand function.

We have $\frac{dD(t)}{dt} = b + 2ct$ and $\frac{d^2D(t)}{dt^2} = 2c$. If $\frac{dD(t)}{dt} = o \Rightarrow t = -\frac{b}{2c}$, which is positive only if b and c are of opposite sign. In this case the demand rate gradually rises to a maximum $\left(a - \frac{b^2}{4c}\right)$ and then gradually decreases. This type of demand is quite appropriate for seasonal products like winter cosmetics. As the season progress, the demand rate begins to rise attains a highest level in the middle and then gone away towards end of the season.

If both b and c are negative, the demand rate $D(t)$ decreases at a deceasing rate which we may call as 'accelerated decline' in demand. This usually happens to the spare parts of an obsolete aircraft model or the microcomputer chip of high technology products which is being substituted by another.

If both b and c are positive, the demand rate $D(t)$ increases at a increasing rate which we may call as 'accelerated growth' in demand which is mainly visible in new computer chips, spare parts of new aeroplane etc.

Thus, we may have different types of realistic demand patterns from the demand rate $D(t) = a + bt + ct^2$ depending on the signs of b and c. Depending on the values of b and c, we have all the types of growth like positive, negative, accelerated decline, accelerated growth which might be suitable to this model. This advantage of the time-quadratic demand has motivated the researchers to adopt it in the present model. Therefore, of the view that quadratic time-dependence demand is more realistic than linear and exponential time-dependence demand.

The proposed model can be extended in several ways. It used in inventory control of certain non-instantaneous deteriorating items such as food items, electronic components, fashionable commodities and others. For instance, we may extend the demand function to stochastic fluctuating demand patterns or stock-dependent demand rate. Finally, we could extend the model to incorporate some more realistic features such as quantity discounts, permissible delay in payments, time value of money, a finite rate of replenishment, inflation, and probabilistic demand etc.

ACKNOWLEDGEMENTS

The authors are indebted to the Editor and the two anonymous reviewers for their detailed comments and valuable suggestions to improve the exposition of this paper.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

Aggarwal, V., Bahari-Kashani, H. (1991). Synchronized production policies for deteriorating items in a declining market. IIE Transaction, 23, 85-197.

Bahari-Kashani, H. (1989). Replenishment schedule for deteriorating items with time-proportional demand. Journal of the Operational Research Society, 40, 75-81.

Begum, R., Sahu, S.K., Sahoo, R.R. (2010). An EOQ model for deteriorating items with Weibull distribution deterioration, unit production cost with quadratic demand and shortages. Applied Mathematical Sciences, 4, 271 – 288.

Chakrabarti, T., Chaudhuri, K.S. (1997). An EOQ model for deteriorating items with a linear trend in demand and shortages in all cycles. International Journal of Production Economics, 49, 205-213.

Chang, H.J., Dye, C.Y. (1999). An EOQ model for deteriorating items with time arying demand and partial backlogging. Journal of the Operational Research Society, 50, 1176–1182.
Chung, K.J., Ting, P.S. (1993). A heuristic for replenishment of deteriorating items with a linear trend in demand. Journal of the Operational Research Society, 44, 1235-1241.
Dave, U. (1989a). A deterministic lot-size inventory model with shortages and a linear trend in demand. Naval Research Logistics, 36, 507-514.
Dave, U. (1989b). On a heuristic inventory replenishment rule for items with a linearly increasing demand incorporating shortages. Journal of the Operational Research Society, 40, 827-830, (1989a).
Dave, U. (1989c). A comment on "An inventory replenishment policy for linearly increasing demand considering shortages – an optimal solution" by Murdeshwar, T.M., Journal of the Operational Research Society, 40, 412-416.
Dave, U., Patel, L.K. (1981). On (T, Si) policy inventory model for deteriorating items with time-proportional demand. Journal of Operational Research Society, 32, 137-142,
Deb, M., Chaudhuri, K.S. (1986). An EOQ model for items with finite rate of production and ariable rate of deterioration. Opsearch, 23, 175-181.
Donaldson, W.A. (1977). Inventory replenishment policy for a linear trend in demand – an analytical solution. Operational Research Quarterly, 28, 663- 670.
Emmons, H. (1968). A replenishment model for radioactive nuclide generators. Management Science, 14, 263-273.
Ghare, P.M., Schrader, G.F. (1963). A model for exponentially decaying inventories. Journal of Industrial Engineering, 14, 238-243.
Ghosh, S.K., Chaudhuri, K.S. (2004). An order-level inventory model for a deteriorating item with Weibull distribution deterioration, time-quadratic demand and shortages. International Journal of Advanced Modelling and Optimization, 6, 31- 45.
Goswami, A., Chaudhuri. K.S. (1991a). EOQ model for an inventory with linear trend in demand and finite rate of replenishment considering shortages. International Journal of Systems Science, 22, 181-187.
Goswami, A., Chaudhuri, K.S. (1991b). An EOQ model for deteriorating items with shortages and a linear trend in demand. Journal of the Operational Research Society, 42, 1105-1110.
Goyal, S.K. (1986). On improving replenishment policies for linear trend in demand. Engineering Costs and Production Economics, 10, 73-76.
Goyal, S.K. (1987). A simple heuristic method for deteriorating economic order interval for linear trend in demand. Engineering Costs and Production Economics, II, 53-57.
Goyal, S.K., Morrin, D., Nebebe, P. (1992). The finite horizon trended inventory replenishment problem with shortages. Journal of the Operational Research Society, 43, 1173-1178.
Goyal, S.K., Giri, B.C. (2001). Recent trends in modeling of deteriorating inventory. European Journal of Operational Research, 134, 1-16.
Hariga, M. (1993). The inventory replenishment problem with continuous linear trend in demand. Computers and Industrial Engineers, 24, 143-150.
Hariga, M. (1996). Optimal EOQ models for deteriorating items with time-varying demand. Journal of the Operational Research Society, 47, 1228-1246.
Hariga, M., Alyan, A. (1997). A lot sizing heuristic for deteriorating items with shortages in growing and declining markets. Computers Operations Research, 24, 1075-1083.
Hariga, M., Benkherouf, L. (1994). Optimal and heuristic inventory replenishment models for deteriorating items with exponential time-varying demand. European Journal of Operational Research, 79, 123-137.

Hollier, R.H., Mak, K.L. (1983). Inventory replenishment policies for deteriorating items in a declining market. International Journal of Production Research, 21, 813-826.

Jalan, A.K., Chaudhuri, K.S. (1999a). Structural properties of an inventory system with deterioration and trended demand. International Journal of Systems Science, 30(6), 627-633.

Jalan, A.K., Chaudhuri, K.S. (1999b). An EOQ model for deteriorating items in a declining market with SFI policy. Korean Journal of Computational and Applied Mathematics, 6(2), 437-449.

Jeyaraman, K., Sugapriya, C. (2008). Determining a common production cycle time for an EPQ model with non-instantaneous deteriorating items allowing price discount using permissible delay in payments. ARPN Journal of Engineering and Applied Sciences, 3, 26-30.

Khanra, S., Chaudhuri, K.S. (2003). A note on order level inventory model for deteriorating item with time-dependent quadratic demand. Computers Operations Research, 30, 1901-1916.

Khanra, S., Sana, S., Chaudhuri, K.S. (2010). An EOQ model for perishable item with stock and price dependent demand rate. International Journal of Mathematics in Operational Research, 2, 320-335.

Kim, K.H. (1995). A heuristic for replenishment of deteriorating items with linear trend in demand. International Journal of Production Economics, 39, 265-270.

Levin, P.I., McLaughlin, C.P., Lamone, R.P., Kottas, J.F. (1972). Contemporary policy for managing operating system. Production Operations Management, McGraw- Hill: New York, 373.

Lin, C., Tan, B., Lee, W.C., (2000). An EOQ model for deteriorating items with time-varying demand and shortages. International Journal of Systems Science, 31, 391-400.

Murdeshwar, T.M. (1988). Inventory replenishment policy for linearly increasing demand considering shortages-an optimal solution. Journal of the Operational Research Society, 39, 687-692.

Ritchie, E. (1980). Practical inventory replenishment policies for a linear trend in demand followed by a period of steady demand. Journal of Operational Research Society, 31, 605-613.

Ritchie, E. (1984). An EOQ for linear increasing demand: a simple optimal solution. Journal of Operational Research Society, 36, 737-739.

Ritchie, E. (1985). Stock replenishment quantities for unbounded linear increasing demand: an interesting consequence of the optimal policy. Journal of Operational Research Society, 31, 605-613.

Roy, M., Sana, S., Chaudhuri, K.S. (2011a). An EOQ model for imperfect quality products with partial backlogging-a comparative study, International Journal of Services and Operations Management, 83-110.

Roy, M., Sana, S., Chaudhuri, K.S. (2011b). An economic order quantity model of imperfect quality items with partial backlogging. International Journal of Systems Science, 42, 1409-1419.

Roy, M., Sana, S., Chaudhuri, K.S. (2011c). An optimal shipment strategy for imperfect items in stock-out situation, Mathematical and Computer Modeling, In Press.

Sachan, R.S. (1984). On (T, S_i) policy inventory model deteriorating items with time-proportional demand. Journal of Operational Research Society, 35, 1013-1019.

Sahu, S.K., Begum, R., Kalam, A. (2007). A note on an optimal replenishment policy for deteriorating items with constant demand and partial backlogging. International Journal of Physical Sciences; Ultra Science, 19(1)M, 65-72.

Sana, S., Chaudhuri, K.S. (2004). On a volume flexible production policy for a deteriorating item with time-dependent demand and shortages. Advanced Modeling and Optimization, 6(1), 57-74.

Sana, S. (2010a). Optimal selling price and lot-size with time varying deterioration and partial backlogging. Applied Mathematics and Computation, 217, 185-194.

Sana, S. (2010b). Demand influenced by enterprises' initiatives — A multi-item EOQ model of deteriorating and ameliorating items. Mathematical and Computer Modeling, 52(1-2), 284-302.

Sana, S. (2011a). The stochastic EOQ model with random sales price. Applied Mathematics and Computation, In Press, Corrected Proof, Available online 21 June.

Sana, S. (2011b). An EOQ model for salesmen's initiatives, stock and price sensitive demand of similar products – A dynamical system. Applied Mathematics and Computation, 218, 3277-3288.

Sana, S. (2011c). An EOQ model of homogeneous products while demand is salesmen's initiatives and stock sensitive. Computers & Mathematics with Applications, 62, 577-587.

Sana, S. An EOQ model for perishable item with stock-dependent demand and price discount rate. American Journal of Mathematical and Management Sciences, (Article in Press).

Shah, Y.K., Jaiswal, M.C. (1977). An order-level inventory model for a system with constant rate of deterioration. Opsearch, 14, 174-184.

Silver. E.A., Meal, H.C. (1969). A simple modification of the EOQ for the case of a varying demand rate. Production of Inventory Management, 10, 52-65.

Wee, H.M. (1995). A deterministic lot size inventory model for deteriorating items with shortages and a declining market. Computers and Operations Research, 22, 345-356.

Whitin, T.M. (1957). Theory of Inventory Management. Princeton University Press, Princeton, NJ, 62-72.

APPENDIX

Convexity condition of the cost function $TVC(t_1,T)$ is established here to ensure the existence of unique point of minimum for $TVC(t_1,T)$.
We have,

$$TVC(t_1,T)=\frac{1}{T}\left[A+c_1\left\{t_d\left[\frac{a}{-}+b\left(\frac{t_1}{-}-\frac{1}{-^2}\right)+c\left(\frac{t_1^2}{-}-\frac{2t_1}{-^2}+\frac{2}{-^3}\right)\right]e^{-(t_1-t_d)}+\left(a\frac{t_d^{\ 2}}{2}+b\frac{t_d^{\ 3}}{3}+c\frac{t_d^{\ 4}}{4}\right)\right.\right.$$

$$-R+P+\left[\frac{a}{-^2}+\frac{b}{-}\left(\frac{t_1}{-}-\frac{1}{-^2}\right)+\frac{c}{-}\left(\frac{t_1^2}{-}-\frac{2t_1}{-^2}+\frac{2}{-^3}\right)\right]\left(e^{-(t_1-t_d)}-1\right)$$

$$\left.-\left[\frac{a}{-}(t_1-t_d)+b\left(\frac{\left(t_1^{\ 2}-t_d^{\ 2}\right)}{2-}-\frac{(t_1-t_d)}{-^2}\right)+c\left(\frac{\left(t_1^{\ 3}-t_d^{\ 3}\right)}{3-}-\frac{\left(t_1^{\ 2}-t_d^{\ 2}\right)}{-^2}+\frac{2(t_1-t_d)}{-^3}\right)\right]\right\}$$

$$+c_2\left\{\left[\frac{a}{-}+b\left(\frac{t_1}{-}-\frac{1}{-^2}\right)+c\left(\frac{t_1^2}{-}-\frac{2t_1}{-^2}+\frac{2}{-^3}\right)\right]e^{-(t_1-t_d)}-M-a(t_1-t_d)-\frac{b}{2}\left(t_1^{\ 2}-t_d^{\ 2}\right)-\frac{c}{3}\left(t_1^{\ 3}-t_d^{\ 3}\right)\right\}$$

$$\left.+N\left[(T-t_1)-\frac{\ln\{1+¢(T-t_1)\}}{¢}\right]\right]$$

$$\frac{\partial TVC(t_1,T)}{\partial t_1}=\frac{1}{T}\left[c_1\left\{t_d\,e^{-(t_1-t_d)}\left[\frac{b}{-}+c\left(\frac{2t_1}{-}-\frac{2}{-^2}\right)\right]+\left(e^{-(t_1-t_d)}-1\right)\left[\frac{b}{-^2}+\frac{c}{-}\left(\frac{2t_1}{-}-\frac{2}{-^2}\right)\right]+\right.\right.$$

$$-.t_d.e^{-(t_1-t_d)}\left[\frac{a}{-}+b\left(\frac{t_1}{-}-\frac{1}{-^2}\right)+c\left(\frac{t_1^{\ 2}}{-}-\frac{2t_1}{-^2}+\frac{2}{-^3}\right)\right]+-.e^{-(t_1-t_d)}\left[\frac{a}{-^2}+\frac{b}{-}\left(\frac{t_1}{-}-\frac{1}{-^2}\right)+\frac{c}{-}\left(\frac{t_1^{\ 2}}{-}-\frac{2t_1}{-^2}+\frac{2}{-^3}\right)\right]$$

$$\left.-\left[\frac{a}{-}+b\left(\frac{t_1}{-}-\frac{1}{-^2}\right)+c\left(\frac{t_1^{\ 2}}{-}-\frac{2t_1}{-^2}+\frac{2}{-^3}\right)\right]\right\}+c_2\left\{-e^{-(t_1-t_d)}\left[\frac{a}{-}+b\left(\frac{t_1}{-}-\frac{1}{-^2}\right)+c\left(\frac{t_1^{\ 2}}{-}-\frac{2t_1}{-^2}+\frac{2}{-^3}\right)\right]\right.$$

$$\left.\left.-\left(a+bt_1+ct_1^{\ 2}\right)+e^{-(t_1-t_d)}\left[\frac{b}{-}+c\left(\frac{2t_1}{-}-\frac{2}{-^2}\right)\right]\right\}+N\left(-1+\frac{1}{[1+¢(T-t_1)]}\right)\right]=0$$

(A1)

$$\frac{\partial^2 TVC(t_1,T)}{\partial t_1^{\,2}} = \frac{1}{T}\Bigg[c_1\Bigg\{\frac{2c}{-}t_d\, e^{-(t_1-t_d)} + \left(e^{-(t_1-t_d)}-1\right)\frac{2c}{-^{2}} + 2^{-}\, t_d e^{-(t_1-t_d)}\left[\frac{b}{-}+c\left(\frac{2t_1}{-}-\frac{2}{-^{2}}\right)\right] +$$

$$2^{-}\, e^{-(t_1-t_d)}\left[\frac{b}{-^{2}}+\frac{c}{-}\left(\frac{2t_1}{-}-\frac{2}{-^{2}}\right)\right] +^{-2}.t_d.e^{-(t_1-t_d)}\left[\frac{a}{-}+b\left(\frac{t_1}{-}-\frac{1}{-^{2}}\right)+c\left(\frac{t_1^{\,2}}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\right)\right]$$

$$+^{-2}.e^{-(t_1-t_d)}\left[\frac{a}{-^{2}}+\frac{b}{-}\left(\frac{t_1}{-}-\frac{1}{-^{2}}\right)+\frac{c}{-}\left(\frac{t_1^{\,2}}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\right)\right]-\left[\frac{b}{-}+c\left(\frac{2t_1}{-}-\frac{2}{-^{2}}\right)\right]\Bigg\}$$

$$+c_2\Bigg\{-b-2ct_1+\frac{2c}{-}e^{-(t_1-t_d)}+2^{-}\, e^{-(t_1-t_d)}\left[\frac{b}{-}+c\left(\frac{2t_1}{-}-\frac{2}{-^{2}}\right)\right]$$

$$+^{-2}e^{-(t_1-t_d)}\left[\frac{a}{-}+b\left(\frac{t_1}{-}-\frac{1}{-^{2}}\right)+c\left(\frac{t_1^{\,2}}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\right)\right]\Bigg\}+\frac{N\text{¢}}{[1+\text{¢}(T-t_1)]^2}\Bigg] > 0$$

(A2)

$$\frac{\partial TVC(t_1,T)}{\partial T} = \frac{N}{T}\left(1-\frac{1}{1+\text{¢}(T-t_1)}\right)-\frac{1}{T^2}\Bigg[A+c_1\Bigg\{t_d e^{-(t_1-t_d)}\left[\frac{a}{-}+b\left(\frac{t_1}{-}-\frac{1}{-^{2}}\right)+c\left(\frac{t_1^{\,2}}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\right)\right]$$

$$-\frac{a}{-}(t_1-t_d)-b\left(\frac{(t_1^{\,2}-t_d^{\,2})}{2^{-}}-\frac{(t_1-t_d)}{-^{2}}\right)-c\left(\frac{2(t_1-t_d)}{-^{3}}-\frac{(t_1^{\,2}-t_d^{\,2})}{-^{2}}+\frac{(t_1^{\,3}-t_d^{\,3})}{3^{-}}\right)$$

$$+\left(e^{-(t_1-t_d)}-1\right)\left[\frac{a}{-^{2}}+\frac{b}{-}\left(\frac{t_1}{-}-\frac{1}{-^{2}}\right)+\frac{c}{-}\left(\frac{t_1^{\,2}}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\right)\right]+P-R\Bigg\}$$

$$+c_2\Bigg\{e^{-(t_1-t_d)}\left[\frac{a}{-}+b\left(\frac{t_1}{-}-\frac{1}{-^{2}}\right)+c\left(\frac{t_1^{\,2}}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\right)\right]-M-a(t_1-t_d)-\frac{b}{2}(t_1^{\,2}-t_d^{\,2})-\frac{c}{3}(t_1^{\,3}-t_d^{\,3})\Bigg\}$$

$$+N\left((T-t_1)-\frac{Log[1+(T-t_1)\text{¢}]}{\text{¢}}\right)\Bigg]=0$$

(A3)

$$\frac{\partial^2 TVC(t_1,T)}{\partial T^2}$$

$$= \frac{N\text{¢}}{T[1+\text{¢}(T-t_1)]^2} - \frac{2N}{T^2}\Big(1-\frac{1}{1+\text{¢}(T-t_1)}\Big) + \frac{2}{T^3}\Bigg[A + c_1\Bigg\{-\frac{a}{-}(t_1-t_d) - b\Bigg(\frac{(t_1^{2}-t_d^{2})}{2-} - \frac{(t_1-t_d)}{-^{2}}\Bigg)$$

$$-c\Bigg(\frac{2(t_1-t_d)}{-^{3}} - \frac{(t_1^{2}-t_d^{2})}{-^{2}} + \frac{(t_1^{3}-t_d^{3})}{3-}\Bigg) + t_d e^{-(t_1-t_d)}\Bigg[\frac{a}{-} + b\Big(\frac{t_1}{-}-\frac{1}{-^{2}}\Big) + c\Big(\frac{t_1^{2}}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\Big)\Bigg]$$

$$+\big(e^{-(t_1-t_d)}-1\big)\Bigg[\frac{a}{-^{2}} + \frac{b}{-}\Big(\frac{t_1}{-}-\frac{1}{-^{2}}\Big) + \frac{c}{-}\Big(\frac{t_1^{2}}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\Big)\Bigg] + P - R\Bigg\}$$

$$+c_2\Bigg\{e^{-(t_1-t_d)}\Bigg[\frac{a}{-} + b\Big(\frac{t_1}{-}-\frac{1}{-^{2}}\Big) + c\Big(\frac{t_1^{2}}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\Big)\Bigg] - M - a(t_1-t_d) - \frac{b}{2}(t_1^{2}-t_d^{2}) - \frac{c}{3}(t_1^{3}-t_d^{3})\Bigg\}$$

$$+N\Bigg((T-t_1) - \frac{Log[1+(T-t_1)\text{¢}]}{\text{¢}}\Bigg)\Bigg] \quad > 0$$

(A4)

$$\frac{\partial^2 TVC(t_1,T)}{\partial T\,\partial t_1} = \frac{\partial^2 TVC(t_1,T)}{\partial t_1 \partial T}$$

$$= -\frac{N\text{¢}}{T[1+\text{¢}(T-t_1)]^2} - \frac{1}{T^2}\Bigg[c_1\Bigg\{t_d\, e^{-(t_1-t_d)}\Bigg[\frac{b}{-} + c\Big(\frac{2t_1}{-}-\frac{2}{-^{2}}\Big)\Bigg] - \frac{a}{-} - b\Big(\frac{t_1}{-}-\frac{1}{-^{2}}\Big) - c\Big(\frac{2}{-^{3}}-\frac{2t_1}{-^{2}}+\frac{t_1^{2}}{-}\Big)$$

$$+t_d - e^{-(t_1-t_d)}\Bigg[\frac{a}{-} + b\Big(\frac{t_1}{-}-\frac{1}{-^{2}}\Big) + c\Big(\frac{t_1^{2}}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\Big)\Bigg] + \big(e^{-(t_1-t_d)}-1\big)\Bigg[\frac{b}{-^{2}} + \frac{c}{-}\Big(\frac{2t_1}{-}-\frac{2}{-^{2}}\Big)\Bigg]$$

$$+- e^{-(t_1-t_d)}\Bigg[\frac{a}{-^{2}} + \frac{b}{-}\Big(\frac{t_1}{-}-\frac{1}{-^{2}}\Big) + \frac{c}{-}\Big(\frac{t_1^{2}}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\Big)\Bigg]\Bigg]\Bigg\} + N\Bigg(-1+\frac{1}{[1+(T-t_1)\text{¢}]}\Bigg)$$

$$+c_2\Bigg\{e^{-(t_1-t_d)}\Bigg[\frac{b}{-} + c\Big(\frac{2t_1}{-}-\frac{2}{-^{2}}\Big)\Bigg] + - e^{-(t_1-t_d)}\Bigg[\frac{a}{-} + b\Big(\frac{t_1}{-}-\frac{1}{-^{2}}\Big) + c\Big(\frac{t_1^{2}}{-}-\frac{2t_1}{-^{2}}+\frac{2}{-^{3}}\Big)\Bigg] - a - bt_1 - ct_1^{2}\Bigg\}\Bigg]$$

(A5)

Now the function $TVC(t_1,T)$ will be convexity if the Hessian determinant,

$$D = \begin{vmatrix} \dfrac{\partial^2 TVC(t_1,T)}{\partial t_1^{2}} & \dfrac{\partial^2 TVC(t_1,T)}{\partial t_1 \partial T} \\ \dfrac{\partial^2 TVC(t_1,T)}{\partial T \partial t_1} & \dfrac{\partial^2 TVC(t_1,T)}{\partial T^{2}} \end{vmatrix} > 0 \qquad (A6)$$

and

$$\frac{\partial^2 TVC(t_1,T)}{\partial T^2} < 0 \tag{A7}$$

where $\frac{\partial^2 TVC(t_1,T)}{\partial t_1^{\ 2}}$, $\frac{\partial^2 TVC(t_1,T)}{\partial T^2}$, $\frac{\partial^2 TVC(t_1,T)}{\partial t_1 \partial T}$ and $\frac{\partial^2 TVC(t_1,T)}{\partial T \partial t_1}$ are given in (A2), (A4) and (A5) respectively. Considering the relation (A4), the condition (A7) is obvious.
