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Modelling Vaccination and Treatment of Childhood Pneumonia and Their Implications

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

This paper presents a deterministic model for pneumonia transmission and uses the model to assess the potential impact of the vaccination, treatment and efficacy of vaccination drugs in lowering the public health impact of the pneumonia disease. The model is based on the Susceptible-Vaccinated-Infected-Treated compartmental classes of children less than five years. There is possibility of the non-severely infected recovering from natural immunity. Model analysis indicates the system lie in the positive region, solution is bounded and there exist unique positive endemic equilibrium point whenever control reproduction number is greater than unity. Important epidemiological thresholds such as the basic and control reproduction number are determined. Disease-free point equilibrium points are determined. Local and Global stability of equilibrium points will be investigated. Sensitivity analysis of the reproduction numbers indicated higher vaccination drug efficacy vaccination, treatment and recoveries from natural immunity hold great promise in lowering pneumonia impact. Estimated numerical result indicated impact of treatment is positive. Numerical simulation was carried to predict the dynamics of the system.

Keywords: Deterministic model; vaccination; sensitivity and simulation.

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1 Introduction

Pneumonia is a common and widespread respiratory disease characterized by an inflammation of the alveoli and is caused by micro-organisms namely: viruses, parasites fungi, and bacteria [1,2]. The leading cause of severe pneumonia among children across the developing world is bacterial pathogen called Streptococcus pneumonia and Haemophilus influenza type b [3].

The symptoms of childhood pneumonia depend on their cause and the age of the infection. Bacterial pneumonia is usually manifested in children by rapid breathing and high fever; it leads to severe pneumonia illness. Viral infections come on slowly over a long time and may worsen over time. Generally, symptoms of pneumonia in children and infants include rapid or difficult breathing, loss of appetite, cough, chills fever, headaches and wheezing. The under five years with severe cases of pneumonia may have difficulty in breathing with their chests moving in or retracting during inhalation (known as 'lower chest wall in drawing'). Young infants may suffer unconsciousness, convulsions, lethargy, hypothermia, and feeding problems [3].

Childhood pneumonia can be protected through: adequate nutrition, exclusive breastfeeding during, access to clean water, regular hand washing, proper sanitation and eliminating indoor air pollution (especially smoke from unsafe cook stoves). Childhood pneumonia can be prevented through: immunization against (pertussis, pneumococcus, Haemophilus influenza type b (Hib) and measles), zinc supplementation for children with diarrhea, antibiotic prophylaxis for HIV-infected children, prevention of HIV infection in children among others [3].

According to UN, reducing the mortality rate for the children under the age of five years by two for every three between 1990 and 2015 was one of the millennium development goals. In 2011, globally, pneumonia was the leading killer of children under the age of five, responsible for nearly one for every five of the worldwide child deaths per year. More than 99 for every one hundred of deaths from pneumonia occurred in the developing world, where access to treatment and healthcare care is often beyond reach for many children [4,5].

In previous mathematical model, bacteria and viruses, being the major causes of the pneumonia are extensively researched and numerous models proposed with assumption that pneumonia was isolated before treatment and in death. This is not the case in most developing countries like Kenya. Fungi and parasitic pneumonia are completely ignored in the developed models. Developing countries have high poverty index resulting to malnourished children and mothers hence no type of pneumonia can be ignored..

Population change in sizes and composition result from: interactions between individuals of the same species, interactions with the environment, interactions between individuals of different species, disease and food supply among others. Interactions can be cooperative, mutualistic, and predatory among others. These changes are expressed in terms of emigration rates, birth rates, death rates and immigration rates. The aims of population dynamics are to understand, explain, and predict the compositions and sizes of populations over time and space. Mathematical modelling is simplification or abstraction of nature, separating the important features of real phenomena from the minor and irrelevant [6].

The research study [7], studied a pneumococcal transmission model which takes into account the risk of higher rates of transmission for children who attend child-care centers. The study assumed children being able to carry only one serotype in a closed community. The results stress the importance of child-care centers in transmission.

The research study [8], considered the issue of coexistence of pneumonia serotypes in a population. The result stressed the importance of correctly modeling the possibility of a host being able to become simultaneously invaded by more than one strain, taking into account difficulties in obtaining a second strain if already colonized and considering acquired immunity of new strains.

The research study [2] considered the bacterial pneumonia with the possibility of temporal immunity, carriers, and treatment. The results stressed importance of treatment and quarantine where possible.

The research studies by [9,10,11] considered the vaccination and treatment of severe and non-severely infected pneumonia in Kenya. The results stressed importance of attaining herd immunity and achieving critical treatments to eradicate childhood pneumonia.

This paper considers specific attributes in developing countries like as no isolations are done in hospitals and possibility of absence of studies have been carried to determine the types of pneumonia which exist in the population. A general model of pneumonia dynamics is developed which takes in account pneumonia immunization effort in the developing countries, treatment and possibility of recovery from natural immunity. This raises important question: based on intervention of vaccination, treatment and recovery from natural immunity, what would be the control reproduction number of childhood pneumonia?

2 Model Formulation

The population is divided in subclasses depending on status of infection and infection characteristics. This model is an improvement of the classical Susceptible-Infected-Susceptible (SIS) model of the population dynamics of infectious disease.

2.1 Model assumptions

This model assumes:

- Homogeneous mixing of the population.
- The type or types of the pneumonia in the population has not been established.
- No isolation of pneumonia in health facilities.
- The study assumes that the treated individuals transmit pneumonia but at lower rates than infected class($\beta_1 > \beta_2$). Treatment reduces the level of infectivity.
- It is also assumed pneumonia induced death is higher in infected class than in treated class($\delta_1 > \delta_2$). Treatment reduces likelihood of dying significantly.
- Although children under the age of five years can contract pneumonia from individuals outside their age bracket (vertical transmission), this study assume their contribution to the force of infection to be not significant.
- Pneumonia infection is assumed to be probably transmitted when less than five years susceptible children come into contact with under five years infected with pneumonia and/or those under five years under treatment that is $\lambda = \beta_1 I + \beta_2 T$.

2.2 Model description

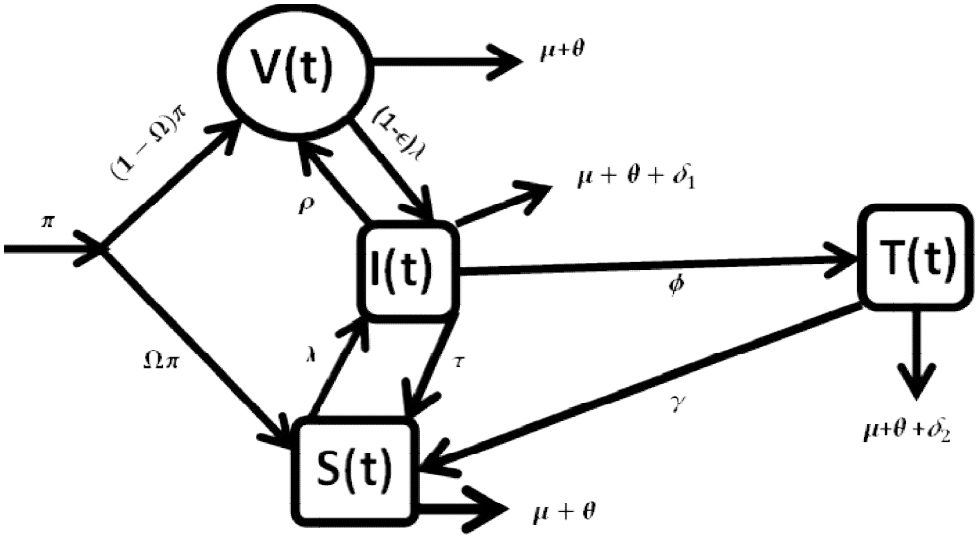
Let N (t) be the total population of the under five years children which is divided into four sub-classes: Susceptible to pneumonia class $S(t)$, the under five years vaccinated with Pneumonia class $V(t)$, the under five years infected with Pneumonia class $I(t)$) and the under five years treated with pneumonia class $T(t)$.

The rate at which infected children seek treatment is given by ϕ. The rate at which treated children recover after treatment is γ. The constant natural death rate is given by μ. The recruitment rate (birth rate) is given by π. The rate at which infected children recover from natural immunity to susceptible and vaccinated classes are τ and ρ respectively. Death due to pneumonia occurs at the rates of δ_1 and δ_2 in infected and treated respectfully. The rate at which children under five progress to the next age class per year is given by θ. The vaccination drug efficacy is given by ε (when $\varepsilon = 1$, the drug is very effective and $\varepsilon = 0$, the drug is useless). The infection rates due to pneumonia in

infected and treated classes are given by β_1 and β_2 respectively. λ (t), is the force of infection (number of individuals who become infected per unit of time),

2.3 Model equations

The above model description can be represented diagrammatically as shown in



From the above flow chart, we obtain the following systems of equations

$$\dot{S} = \Omega\pi - (\lambda + \theta + \mu)S + \gamma T + \tau I \quad (1),$$

$$\dot{V} = (1-\Omega)\pi - [\theta + \mu + (1-\varepsilon)\lambda]V + \rho I \quad (2),$$

$$\dot{I} = \lambda S + (1-\varepsilon)\lambda V - \omega_1 I \quad (3),$$

$$\dot{T} = \phi I - \omega_2 T \quad (4),$$

Where

$$N(t) = S(t) + V(t) + I(t) + T(t) \quad (5),$$

$$\lambda = \beta_1 I + \beta_2 T \quad (6),$$

$$\omega_1 = \phi + \theta + \mu + \tau + \rho + \delta_1 \quad (7),$$

$$\omega_2 = \gamma + \theta + \mu + \delta_2 \quad (8),$$

$$0 \leq \Omega \leq 1 \quad (9).$$

We obtain the total time derivative of the total population by adding system of the equations $[(1)-(4)]$ to obtain,

$$\dot{N} = \pi - (\theta + \mu)N - \delta_1 I - \delta_2 T,$$

3 Model Analysis

3.1 Positivity and the boundness of the solutions

Theorem 1.

The region R given by

$$R = \left\{[S(t), V(t), I(t) \text{and } T(t)] \epsilon R_+^4 \,|S(0) \geq 0, V(0) \geq 0, I(0) \geq 0, T(0) \geq 0 \text{ and } N(t) \leq \frac{\pi}{\mu + \theta}\right\}$$

is positively invariant and attracting with respect to the system of equations (1) - (5).

Proof

Let S, V, I and T be any solution of the system with non-negative initial conditions.

Since $\dot{S} = \Omega\pi - (\lambda + \theta + \mu)S + \gamma T + \tau I$, it follows that $\frac{d}{dt}[S(t)\, e^{\int_o^t \{\lambda(s)+\theta+\mu\} ds}] \geq 0$, hence $S(t)\, e^{\int_o^t \{\lambda(s)+\theta+\mu\}\, ds}$ is a non-negative function of t, thus S(t) stays positive.

From $\dot{V} = (I - \Omega)\pi - [\theta + \mu + (1 - \varepsilon)\lambda]V + \rho I$, $\dot{V} > -[\theta + \mu + (1 - \varepsilon)\lambda]V$, which implies that $V(t) > V(0)\, e^{-\theta+\mu+1-\varepsilon\lambda t} \geq 0$. It follows that $I(t) > I(0)\, e^{-\omega 1 t} \geq 0$. It also follows that $T(t) > T(0)\, e^{-\omega 2 t} \geq 0$.

For boundedness of solution we take the time derivative of our total population along its

solution to obtain $\dot{N(t)} = \pi - (\mu + \theta)N(t) - \delta_1 I(t) - \delta_2 T(t)$. Now,

$$\dot{N(t)} + (\mu + \theta)N = \pi - \delta_1 I(t) - \delta_2 T(t),$$
$$\dot{N(t)} + (\mu + \theta)N(t) \leq \pi$$

So that

$$N(t) \leq \frac{\pi}{\mu+\theta}\,(1 + Ce^{-(\mu+\theta)t})$$

Where C is a constant of integration

$$\lim_{t\to\infty} N(t) \leq \frac{\pi}{\mu + \theta}$$

This proves the boundedness of the solutions inside R. This implies that all solutions of the System (1) - (5) starting in R remain in R for all time $t \geq 0$. Thus R is positively invariant and attracting and hence it is sufficient to consider the dynamics of our system in R [8, 12].

3.2 Equilibrium points and reproduction number

3.2.1 Disease-free equilibrium point (DFE)

Let $E^0 = (S^0, V^0, I^0 \text{and } T^0)$ be the disease-free point. Then E^0 of the system (1) – (5), is obtained by setting all the infectious classes and treatment classes to zero. We get

$$\Omega\pi - (\theta + \mu)S^0 = 0$$

which yields

$$S^0 = \frac{\Omega\pi}{(\theta+\mu)}$$

Also

$$(1-\Omega)\pi - [\theta+\mu]V^0 = 0$$
$$V^0 = \frac{(1-\Omega)\pi}{(\theta+\mu)}$$

The DFE point for our system is given by,

$$E^0 = (S^0, V^0, I^0 \text{and } T^0) = \left(\frac{\Omega\pi}{(\theta+\mu)}, \frac{(1-\Omega)\pi}{(\theta+\mu)}, 0, 0\right) \quad (2)$$

3.2.2 The basic reproduction number R_0 and control reproduction number R_C.

We use the next-generation matrix method to determine the basic reproduction number, R_0, and control reproduction number(R_C) of the model (2). Using the notation f for a matrix of new infections terms and $\surd$ for the matrix of the remaining transfer terms in our system, we get,

$$f = \begin{pmatrix} \lambda S + (1-\varepsilon)\lambda V \\ 0 \end{pmatrix}, \surd = \begin{pmatrix} \omega_1 I \\ \omega_2 T - \phi I \end{pmatrix}.$$

We obtain the matrices F and V by finding the Jacobian matrices of f and $\surd$evaluated at DFE as follows

$$F = \begin{pmatrix} \beta_1 S^0 + \beta_1(1-\varepsilon)V^0 & \beta_2 S^0 + \beta_2(1-\varepsilon)V^0 \\ 0 & 0 \end{pmatrix}; V = \begin{pmatrix} \omega_1 & 0 \\ -\phi & \omega_2 \end{pmatrix}$$

Now we compute the inverse of matrix V to obtain

$$V^{-1} = \begin{pmatrix} \frac{1}{\omega_1} & 0 \\ \frac{\phi}{\omega_1\omega_2} & \frac{1}{\omega_2} \end{pmatrix}$$

,

Using Mathematica software we obtain the eigenvalues(η_i) of the matrix FV^{-1} as,

$$\eta_1 = 0,$$
$$\eta_2 = \frac{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0}{\omega_1} + \frac{\phi\{\beta_2 S^0 + \beta_2(1-\varepsilon)V^0\}}{\omega_1\omega_2}.$$

The control reproduction number (R_C)is given by the spectral radius ζ (the dominant eigenvalue) of the matrix FV^{-1}, denoted by $\zeta(FV^{-1})$. The eigenvalue η_2 is the spectral radius.

The control reproduction number (R_C) is the average number of secondary infections one infectious individual (either in infectious or treated class) can infect when combined interventions of treatment, recovery from natural immunity and immunization are in place is expressed as,

$$R_C = \frac{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0}{\omega_1} + \frac{\phi\{\beta_2 S^0 + \beta_2(1-\varepsilon)V^0\}}{\omega_1\omega_2}$$

The other reproduction numbers are derived from the control reproduction number(R_C) by varying various parameters as below;

i. When there are no recoveries from natural immunity of the infected individuals to susceptible class, the corresponding reproduction is obtained by setting the parameter(τ) to zero in the expression of R_C, to obtain,

$$R_1 = \frac{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0}{\phi + \theta + \mu + \rho + \delta_1} + \frac{\phi\{\beta_2 S^0 + \beta_2(1-\varepsilon)V^0\}}{(\phi + \theta + \mu + \rho + \delta_1)\omega_2}$$

ii. When there are no recoveries from natural immunity of the infected individuals to vaccinated class, the corresponding reproduction number is obtained by setting the parameter(ρ) to zero in the expression of R_C, to obtain,

$$R_2 = \frac{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0}{\phi + \theta + \mu + \tau + \delta_1} + \frac{\phi\{\beta_2 S^0 + \beta_2(1-\varepsilon)V^0\}}{(\phi + \theta + \mu + \tau + \delta_1)\omega_2}$$

iii. When there are no recoveries from natural immunity of the infected individuals to vaccinated class and susceptible class, the corresponding reproduction number is obtained by setting the parameters(ρ, τ) to zero in the expression of R_C, to obtain,

$$R_3 = \frac{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0}{\phi + \theta + \mu + \delta_1} + \frac{\phi\{\beta_2 S^0 + \beta_2(1-\varepsilon)V^0\}}{(\phi + \theta + \mu + \delta_1)\omega_2}$$

iv. When there is no treatment infected individuals, the corresponding reproduction number is obtained by setting the parameter(ϕ) to zero in the expression of R_C, to obtain,

$$R_4 = \frac{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0}{\theta + \mu + \tau + \rho + \delta_1}$$

v. When the vaccination drug efficacy is 100% efficient, the corresponding reproduction number is obtained by setting the parameter(ε) to one in the expression of R_C, to obtain,

$$R_5 = \frac{\beta_1 S^0}{\omega_1} + \frac{\phi\beta_2 S^0}{\omega_1\omega_2}$$

vi. When the vaccination drug efficacy is ineffective(useless), the corresponding reproduction number is obtained by setting the parameter(ε) to zero in the expression of R_C, to obtain,

$$R_6 = \frac{\beta_1 S^0 + \beta_1 V^0}{\omega_1} + \frac{\phi\{\beta_2 S^0 + \beta_2 V^0\}}{\omega_1\omega_2}$$

vii. In absence of any interventions(immunizations, recoveries from natural immunity and treatment), the corresponding basic reproduction number (R_0) is obtained by setting the parameters ($\varepsilon, \phi,$ ρ and,) to zero in the expression of R_C, to obtain,

$$R_0 = \frac{\beta_1(S^0 + V^0)}{\theta + \mu + \delta_1}$$

3.3 Existence of the Endemic equilibrium point (EEP)

Theorem 2

A positive endemic equilibrium exist if and only if $R_C > 1$.
Proof

Let $E^* = (S^*, V^*, I^* \text{ and } T^*)$ be the disease free point. Then E^* of the system (1) – (5), is obtained by setting all the classes to zero. We obtain the following systems of equations;

$$\Omega\pi - (\lambda^* + \theta + \mu)S^* + \gamma T^* + \tau I^* = 0 \quad \text{(i)}$$

$$(1-\Omega)\pi - [\theta + \mu + (1-\varepsilon)\lambda^*]V^* + \rho I^* = 0 \quad \text{(ii)}$$

$$\lambda^* S^* + (1-\varepsilon)\lambda^* V^* - \omega_1 I^* = 0 \quad \text{(iii)}$$

$$\phi I^* - \omega_2 T^* = 0 \quad \text{(iv)}$$

Also at EEP the force of infection λ^* is given by

$$\lambda^* = \beta_1 I^* + \beta_2 T^* \quad \text{(v)}$$

Solving the system of equation above (i) – (iv) in terms of λ^* using Mathematica software we obtain;

$$\begin{aligned} S^* = {} & (\pi\big((-1+\varepsilon)(\gamma\phi(-1+\Omega) + (-\tau + (\rho+\tau)\Omega)\omega_2)\lambda^* + \Omega\omega_1\omega_2(\theta+\mu+\lambda^* - \varepsilon\lambda^*)\big)) \\ & /(\omega_1\omega_2(\theta+\mu+\lambda^*)(\theta+\mu+\lambda^*-\varepsilon\lambda^*) \\ & + \lambda^*(-(\theta+\mu)(\gamma\phi + (\rho - \varepsilon\rho + \tau)\omega_2) + (-1+\varepsilon)(\gamma\phi + (\rho+\tau)\omega_2)\lambda^*)) \\ V^* = {} & \frac{1}{\theta+\mu+\lambda^*-\varepsilon\lambda^*}(\pi - \pi\Omega \\ & + (\Omega\rho\omega_2\lambda^*\left((\theta+\mu)\big(1+\varepsilon(-1+\Omega)\big) - (-1+\varepsilon)\lambda^*\right)) \\ & /(\omega_1\omega_2(\theta+\mu+\lambda^*)(\theta+\mu+\lambda^*-\varepsilon\lambda^*) \\ & + \lambda^*(-(\theta+\mu)(\gamma\phi + (\rho - \varepsilon\rho + \tau)\omega_2) + (-1+\varepsilon)(\gamma\phi + (\rho+\tau)\omega_2)\lambda^*))) \\ I^* = {} & (\pi\lambda^*\big(-(\theta+\mu)\big(1+\varepsilon(-1+\Omega)\big) + (-1+\varepsilon)\lambda^*\big)) \\ & /((\theta+\mu+\lambda^*)(\theta+\mu+\lambda^*-\varepsilon\lambda^*)\left(-\omega_1 + \lambda^*\left(\frac{\gamma\phi+\tau\omega_2}{\omega_2(\theta+\mu+\lambda^*)} + \frac{\rho-\varepsilon\rho}{\theta+\mu+\lambda^*-\varepsilon\lambda^*}\right)\right)) \\ T^* = {} & (\pi\phi\lambda^*\big(-(\theta+\mu)\big(1+\varepsilon(-1+\Omega)\big) + (-1+\varepsilon)\lambda^*\big)) \\ & /(\omega_2(\theta+\mu+\lambda^*)(\theta+\mu+\lambda^*-\varepsilon\lambda^*)\left(-\omega_1 + \lambda^*\left(\frac{\gamma\phi+\tau\omega_2}{\omega_2(\theta+\mu+\lambda^*)} + \frac{\rho-\varepsilon\rho}{\theta+\mu+\lambda^*-\varepsilon\lambda^*}\right)\right)) \end{aligned}$$

Substituting T^* in (v) using equation (iv) we obtain

$$\lambda^* = \frac{\omega_1 R_C}{S^0 + (1-\varepsilon)V^0} I^* \quad \text{(vi)}$$

Substituting I^* (obtained in terms λ^*) in equation (vi) using Mathematica Software to obtain

$$\begin{aligned} \lambda^* = {} & (\pi R_C\omega_1\lambda^*(-(\theta+\mu)(1+\varepsilon(-1+\Omega) + (-1+\varepsilon)\lambda^*))/(S^0 + V^0 - \varepsilon V^0)(\theta+\mu+\lambda^*)(\theta+\mu+\lambda^* \\ & - \varepsilon\lambda^*)(-\omega_1 + \lambda^*\left(\frac{\gamma\phi+\tau\omega_2}{\omega_2(\theta+\mu+\lambda^*)} + \frac{\rho-\varepsilon\rho}{\theta+\mu+\lambda^*-\varepsilon\lambda^*}\right))) \end{aligned}$$

Case 1

$\lambda^* = 0$, this corresponds to disease free equilibrium point (DFE)

Case 2

$$1 = \left(\frac{\pi R_C \omega_1 \left(-(\theta+\mu)(1+\varepsilon(-1+\Omega)+(-1+\varepsilon)\lambda^*)\right)}{S^0+V^0-\varepsilon V^0(\theta+\mu+\lambda^*)(\theta+\mu+\lambda^*-\varepsilon\lambda^*)\left(-\omega_1+\lambda^*\left(\frac{\gamma\phi+\tau\omega_2}{\omega_2(\theta+\mu+\lambda^*)}+\frac{\rho-\varepsilon\rho}{\theta+\mu+\lambda^*-\varepsilon\lambda^*}\right)\right)} \right),$$

which corresponds to endemic equilibrium point (EEP) that is $E^* = (S^*, V^*, I^* \text{ and } T^*)$. Solving the equation above for λ^* using Mathematica software to obtain a quadratic equation of the form $\lambda^{**^2} + b_1\lambda^{**} + b_2 = 0$, whose solution is $\lambda^{**} = \frac{-b_1 \pm \sqrt{(b_1{}^2 - 4b_2)}}{2}$.

For positive value of λ^{**}; $-b_1 + \sqrt{(b_1{}^2 - 4b_2)} > 0$ and $-b_1 - \sqrt{(b_1{}^2 - 4b_2)} > 0$ which implies $b_2 < 0$. Using Mathematica software,

$$\begin{aligned} b_1 = &-(\gamma(S^0+V^0-\varepsilon V^0)(\theta+\mu)\phi \\ &+\left(-(S^0+V^0-\varepsilon V^0)(\theta+\mu)\left((-1+\varepsilon)\rho-\tau\right)\right. \\ &+\left((-2+\varepsilon)(S^0+V^0-\varepsilon V^0)(\theta+\mu)-\pi(-1+\varepsilon)R_C\right)\omega_1\Big)\omega_2) \\ &/((-1+\varepsilon)(S^0+V^0-\varepsilon V^0)(\gamma\phi+(\rho+\tau-\omega_1)\omega_2)) \end{aligned}$$

$$b_2 = \frac{(\theta+\mu)((S^0+V^0-\varepsilon V^0)(\theta+\mu)+\pi(-1+\varepsilon(1-\Omega))R_C)\omega_1\omega_2}{(-1+\varepsilon)(S^0+V^0-\varepsilon V^0)(\gamma\phi+(\rho+\tau-\omega_1)\omega_2)}$$

$$b_2 = \frac{(\theta+\mu)((S^0+V^0-\varepsilon V^0)(\theta+\mu)+\pi(-1+\varepsilon(1-\Omega))R_C)\omega_1\omega_2}{(-1+\varepsilon)(S^0+V^0-\varepsilon V^0)\left((-\phi\ -\theta-\mu-\delta_1)(\theta+\mu+\delta_2)\right)}$$

Clearly, $(-1+\varepsilon)(S^0+V^0-\varepsilon V^0)\left((-\phi\ -\theta-\mu-\delta_1)(\theta+\mu+\delta_2)\right) \geq 0$. The necessary and sufficient condition for positive endemic equilibrium point is

$$(\theta+\mu)\omega_1\omega_2((S^0+V^0-\varepsilon V^0)(\theta+\mu)+\pi(-1+\varepsilon(1-\Omega))R_C) < 0,$$

Since $\{(\theta+\mu)\,\omega_1\omega_2 > 0, it\ imples\ ((S^0+V^0-\varepsilon V^0)(\theta+\mu)+\pi\left(-1+\varepsilon(1-\Omega)\right)R_C) < 0$, Substituting S^0 and V^0 we obtain,

$\pi\left(1-\varepsilon(1-\Omega)\right)-R_C\pi\left(1-\varepsilon(1-\Omega)\right) < 0$, this implies $R_C > 1$, this completes the proof.

Numerical simulation of the full model in Fig. 4 confirms this.

3.4 Bifurcation analysis

The possible presence of two endemic equilibriums above indicates the possibility of bifurcation in the model. Our research study explored it using the Centre Manifold theory. To apply this theory, the following simplification and change of variables are made. Let $S = x_1, V = x_2, I = x_3$ and $T = x_4$, so that $N = x_1 + x_2 + x_3 + x_4$. Further, by using vector notation

$x = (x_1, x_2, x_3, x_4)^T$, the model $[(1)-(4)]$ can be written in the form $\frac{dx}{dt} = F(x)$, with $F = (f_1, f_2, f_3, f_4)^T$, as follows:

$$\dot{x_1} = f_1 = \Omega\pi - (\lambda + \theta + \mu)x_1 + \tau x_3 + \gamma x_4 \quad (4.4.1),$$

$$\dot{x_2} = f_2 = (1-\Omega)\pi - [\theta + \mu + (1-\varepsilon)\lambda]x_2 + \rho x_3 \quad (4.4.2),$$

$$\dot{x_3} = f_3 = \lambda x_1 + (1-\varepsilon)\lambda x_2 - \omega_1 x_3 \quad (4.4.3),$$

$$\dot{x_4} = f_4 = \phi x_3 - \omega_2 x_4 \quad (4.4.4),$$

with, $\lambda = \beta^* x_3 + \beta_2 x_4$.

Evaluating the Jacobian of the system $[(4.4.1) - (4.4.4)]$ at the disease-free equilibrium point(DFE), denoted by $J(E_*^0)$, we obtain,

$$J(E_*^0). = \begin{pmatrix} -(\theta+\mu) & 0 & -\beta^* S^0 + \tau & -\beta_2 S^0 + \gamma \\ 0 & -(\theta+\mu) & -(1-\varepsilon)\beta^* V^0 + \rho & -(1-\varepsilon)\beta_2 V^0 \\ 0 & 0 & \beta^* S^0 + (1-\varepsilon)\beta^* V^0 - \omega_1 & \beta_2 S^0 + (1-\varepsilon)\beta_2 V^0 \\ 0 & 0 & \phi & -\omega_2 \end{pmatrix}.$$

,

Consider a case when $R_C = 1$. Let $\beta_1 = \beta^*$ be bifurcation parameter. Solving for β_1 from $R_C = 1$, we obtain,

$$\beta^* = \beta_1 = \frac{\omega_1}{S^0 + (1-\varepsilon)V^0} - \frac{\phi\beta_2}{\omega_2}.$$

Using Mathematica software, it follows that the Jacobian of $\frac{dx}{dt} = F(x)$ at the DFE, with $\beta_1 = \beta^*$, denoted by $J(E_*^0)$, has eigenvalues $\{0, -(\theta+\mu), -(\theta+\mu)$ and $-\frac{\phi\beta_2(S^0+(1-\varepsilon)V^0+\omega_2^2}{\omega_2}\}$.

The presence of a simple zero eigenvalue (with all other eigenvalues having negative real part), make it viable for the Centre Manifold theory can be used to analyze the dynamics of the model.

Theorem 3. Castillo-Chavez and Song. Consider the following general system of ordinary differential equations with a parameter β^*

$$\frac{dx}{dt} = f(x, \beta^*), f: R^n \times R \to R^n \text{ and } f \in C^2(R^n \times R),$$

where 0 is an equilibrium point of the system (that is, $f(x, \beta^*) \equiv 0$ for all φ) and

1. $A = D_x f(0,0) = \left(\frac{\delta f_i}{\delta x_j}(0,0)\right)$ is the linearization matrix of the system aroundthe equilibrium 0 with φ evaluated at 0;
2. Zero is a simple eigenvalue of A and all other eigenvalues of A have negative real parts;
3. Matrix A has a right eigenvector w and a left eigenvector v corresponding to the zero eigenvalue.

Let f_k be the kth component of f and

$$a = \sum_{k,ij=1}^{n} v_k u_i u_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0),$$

$$b = \sum_{k,ij=1}^{n} v_k u_i \frac{\partial^2 f_k}{\partial x_i \partial \beta^*}(0,0).$$

then the local dynamics of the system around the equilibrium point 0 is determined by the signs of a and b. Particularly when:

i. $a > 0\ and\ b > 0$, when $\beta^* < 0$ with $|\beta^*| \ll 1, (0,0)$, is locally asymptotically stable and there exists a positive unstable equilibrium; when $0 < \beta^* \ll 1, (0,0)$ is unstable, and there exists a negative and locally asymptotically stable equilibrium.

ii. $a < 0\ and\ b < 0$, when $\beta^* < 0$ with $|\beta^*| \ll 1, (0,0)$ is unstable; when $0 < \beta^* \ll 1, (0,0)$ is asymptotically stable and there exists a positive unstable equilibrium.

iii. $a < 0\ and\ b > 0$, when $\beta^* < 0$ with $|\beta^*| \ll 1, (0,0)$ is unstable, and there exists a negative and locally asymptotically stable equilibrium; when $0 < \beta^* \ll 1, (0,0)$ is stable and there exists a positive unstable equilibrium.

iv. $a > 0\ and\ b < 0$, when β^* changes from negative to positive, $(0,0)$ changes its stability from stable to unstable. Correspondingly a negative equilibrium becomes positive and locally asymptotically stable.

If a > 0 and b > 0, then the system has backward bifurcation.

Eigenvectors of J_{β^*}: For the case when $R_C = 1$, it can be shown that the Jacobian of …at $\beta = \beta^*$ (denoted by J_{β^*}) has a right eigenvector given by $u = [u_1, u_2, u_3, u_4]^T$, where,

$$u_1 = -\frac{(\beta^* S^0 - \tau)u_3 + (\beta_2 S^0 - \gamma)u_4}{(\theta + \mu)} < 0,$$

$$u_2 = -\frac{\{(1-\varepsilon)\beta_1 V^0 - \rho\}u_3 + \{(1-\varepsilon)\beta_2 V^0\}u_4}{(\theta + \mu)} < 0,$$

$$u_3 = u_3 > 0 \text{ and } u_4 = \frac{\phi u_3}{\omega_2} > 0.$$

Further, J_{β^*} has a left eigenvectors $v = [v_1, v_2, v_3, v_4]^T$, where,

$$J(E_*^0). = \begin{pmatrix} -(\theta+\mu) & 0 & -\beta^* S^0 + \tau & -\beta_2 S^0 + \gamma \\ 0 & -(\theta+\mu) & -(1-\varepsilon)\beta^* V^0 + \rho & -(1-\varepsilon)\beta_2 V^0 \\ 0 & 0 & \beta^* S^0 + (1-\varepsilon)\beta^* V^0 - \omega_1 & \beta_2 S^0 + (1-\varepsilon)\beta_2 V^0 \\ 0 & 0 & \phi & -\omega_2 \end{pmatrix}$$

$$v_1 = 0, v_2 = 0, v_3 = v_3 > 0 \text{ and } v_4 = \frac{\{(1-\varepsilon)\beta_2 V^0\}v_3}{\omega_2} > 0.$$

Since $v_1 = v_2 = 0$, we only need to compute the partial derivatives of f_3, f_4 (at the DFE). For the system, the associated non-zero partial derivative of f_3, f_4 (at the DFE) is given by

$$\frac{\partial^2 f_3}{\partial x_1 \partial x_3} = \frac{\partial^2 f_3}{\partial x_3 \partial x_1} = \beta^*, \quad \frac{\partial^2 f_3}{\partial x_1 \partial x_4} = \frac{\partial^2 f_3}{\partial x_4 \partial x_1} = \beta_2, \quad \frac{\partial^2 f_3}{\partial x_2 \partial x_3} = \frac{\partial^2 f_3}{\partial x_3 \partial x_2} = (1-\varepsilon)\beta^*,$$
$$\frac{\partial^2 f_3}{\partial x_2 \partial x_4} = \frac{\partial^2 f_3}{\partial x_4 \partial x_2} = (1-\varepsilon)\beta_2.$$

It implies,

$$a = v_3 \sum_{i,j=1}^{4} u_i u_j \frac{\partial^2 f_k}{\partial x_i \partial x_j} + v_4 \sum_{i,j=1}^{4} u_i u_j \frac{\partial^2 f_k}{\partial x_i \partial x_j},$$

$$a = v_3\left\{u_1u_3\frac{\partial^2 f_3}{\partial x_1\,\partial x_3} + u_3u_1\frac{\partial^2 f_3}{\partial x_3\,\partial x_1} + u_1u_4\frac{\partial^2 f_3}{\partial x_1\,\partial x_4} + u_4u_1\frac{\partial^2 f_3}{\partial x_4\,\partial x_1} + u_2u_3\frac{\partial^2 f_3}{\partial x_2\,\partial x_3} + u_3u_2\frac{\partial^2 f_3}{\partial x_3\,\partial x_2} + u_2u_4\frac{\partial^2 f_3}{\partial x_2\,\partial x_4} + u_4u_2\frac{\partial^2 f_3}{\partial x_4\,\partial x_2}\right\}$$
$$+ v_4\left\{u_1u_3\frac{\partial^2 f_3}{\partial x_1\,\partial x_3} + u_3u_1\frac{\partial^2 f_3}{\partial x_3\,\partial x_1} + u_1u_4\frac{\partial^2 f_3}{\partial x_1\,\partial x_4} + u_4u_1\frac{\partial^2 f_3}{\partial x_4\,\partial x_1} + u_2u_3\frac{\partial^2 f_3}{\partial x_2\,\partial x_3} + u_3u_2\frac{\partial^2 f_3}{\partial x_3\,\partial x_2} + u_2u_4\frac{\partial^2 f_3}{\partial x_2\,\partial x_4} + u_4u_2\frac{\partial^2 f_3}{\partial x_4\,\partial x_2}\right\},$$
$$= 2v_3\{u_1u_3\beta^* + u_1u_4\beta_2 + u_2u_3(1-\varepsilon)\beta^* + u_2u_4(1-\varepsilon)\beta_2\}$$
$$+ 2v_4\{u_1u_3\beta^* + u_1u_4\beta_2 + u_2u_3(1-\varepsilon)\beta^* + u_2u_4(1-\varepsilon)\beta_2\} < 0,$$

Also,

$$\frac{\partial^2 f_3}{\partial x_3\,\partial\beta^*} = S^0 + (1-\varepsilon)V^0,$$
$$b = v_3\sum_{i,j=1}^{4} u_i\frac{\partial^2 f_k}{\partial x_i\,\partial\beta^*} + v_4\sum_{i,j=1}^{4} u_i\frac{\partial^2 f_k}{\partial x_i\,\partial\beta^*},$$
$$b = v_3u_3\frac{\partial^2 f_3}{\partial x_3\,\partial\beta^*} + v_4u_3\frac{\partial^2 f_3}{\partial x_3\,\partial\beta^*} > 0$$

Hence, it follows $(a < 0\ and\ b > 0)$, when $\beta^* < 0$ with $|\beta^*| \ll 1, (0,0)$ is unstable, and there exists a negative and locally asymptotically stable equilibrium; when $0 < \beta^* \ll 1, (0,0)$ is stable and there exists a positive unstable equilibrium..

3.5 Local Stability of the DFE point

Theorem 4.

The DFE of the system (1) - (4) is locally asymptotically stable if and only if $R_C < 1$.

Proof

To establish the local stability of E^o, we use the Jacobian of the model evaluated at E^o. Security of this steady state is then determined based on the eigenvalues of the corresponding Jacobian which are functions of the model parameters.

From the system $[(1)-(4)]$ we let

$$F_1 = \Omega\pi - (\lambda+\theta+\mu)S + \gamma T + \tau I, \qquad F_2 = (1-\Omega)\pi - [\theta+\mu+(1-\varepsilon)\lambda]V + \rho I,$$
$$F_3 = \lambda S + (1-\varepsilon)\lambda V - \omega_1 I \text{ and } F_4 = \phi I - \omega_2 T$$

$$J = \begin{pmatrix} \frac{dF_1}{dS} & \frac{dF_1}{dV} & \frac{dF_1}{dI} & \frac{dF_1}{dT} \\ \frac{dF_2}{dS} & \frac{dF_2}{dV} & \frac{dF_2}{dI} & \frac{dF_2}{dT} \\ \frac{dF_3}{dS} & \frac{dF_3}{dV} & \frac{dF_3}{dI} & \frac{dF_3}{dT} \\ \frac{dF_4}{dS} & \frac{dF_4}{dV} & \frac{dF_4}{dI} & \frac{dF_4}{dT} \end{pmatrix},$$

We obtain

$$J(E^0) = \begin{pmatrix} -(\theta+\mu) & 0 & -\beta_1 S^0 + \tau & -\beta_2 S^0 + \gamma \\ 0 & -(\theta+\mu) & -((1-\varepsilon)\beta_1 V^0) + \rho & -((1-\varepsilon)\beta_2 V^0) \\ 0 & 0 & \beta_1 S^0 + ((1-\varepsilon)\beta_1 V^0) - \omega_1 & \beta_2 S^0 + ((1-\varepsilon)\beta_2 V^0) \\ 0 & 0 & \phi & -\omega_2 \end{pmatrix}$$

We obtain the eigenvalues $q(i)$, where $i = 1(1)4$ of $J(E^0)$ using Mathematica software as,

$$q(1) = q(2) = -(\theta+\mu);\ q(3) = \frac{1}{2}\left(\left(S^0 + V^0(1-\varepsilon)\right)\beta_1 - \omega_1 - \omega_2 - \sqrt{u}\right)$$
$$q(4) = \frac{1}{2}\left(\left(S^0 + V^0(1-\varepsilon)\right)\beta_1 - (\omega_1 + \omega_{2)} + \sqrt{u}\right)$$

Where,

$$u = \left\{\left(S^0 + V^0(1-\varepsilon)\right)^2\beta_1^2 + 4\left(S^0 + V^0(1-\varepsilon)\right)\phi\beta_2 + (\omega_1 - \omega_2)^2 + 2\left(S^0 + V^0(1-\varepsilon)\right)\beta_1(-\omega_1 + \omega_2)\right\}$$

Clearly, $q(1) = q(2) < 0\ and\ q(3) < 0$. By inspection,

$$\sqrt{u} > \left(\left(S^0 + V^0(1-\varepsilon)\right)\beta_1 - \omega_1 - \omega_2\right)$$

After algebraic evaluation the conditions necessary for $q(4) < 0$ is $\frac{\left(S^0+V^0(1-\varepsilon)\right)\beta_1}{\omega_1} + \frac{\left(S^0+V^0(1-\varepsilon)\right)\phi\beta_2}{\omega_1\omega_2} < 1$. This implies $R_C < 1$ which completes the proof.

3.6 Global stability of the DFE point

Theorem 5

The DFE is globally stable

Proof

To establish the global asymptotic stability of the disease-free equilibrium point, E^0, the method by Castillo-Chavez is used. The system can $[(1) - (4)]$ be rewritten as:

$$\frac{dX}{dt} = F(X, Z);\ \frac{dZ}{dt} = G(X, Z), G(X, 0) = 0$$

Here $X = (S, V)$ represents the uninfected classes, while $Z = (I, T)$ represents the infected and treated classes. $E^0 = (X^*; 0) = \left(\frac{\Omega\pi}{(\theta+\mu)}, \frac{(1-\Omega)\pi}{(\theta+\mu)}, 0, 0\right)$, denotes the disease-free equilibrium point of system (1) above. To guarantee global asymptotic stability, conditions (H1) and (H2) below must be satisfied.

(H1) For $\frac{dX}{dt} = F(X, 0), X*$ is globally asymptotically stable (g. a. s),
(H2) $G(X, Z) = AZ - \hat{G}(X, Z)$, $\hat{G}(X, Z) \geq 0$ for $(X, Z) \in R^4_{+,}$, where $= D_Z G(X^*, 0)$ is an M matrix and $R^3_{+,}$, is the region where the model makes biological sense. In this case

$$F(X,0)=\begin{pmatrix}\Omega\pi-(\theta+\mu)S\\(1-\Omega)\pi-(\theta+\mu)V\end{pmatrix},\ A=\begin{bmatrix}\beta_1S^0+(1-\varepsilon)\beta_1V^0-\omega_1 & \beta_2S^0+(1-\varepsilon)\beta_2V^0\\ \phi & -\omega_2\end{bmatrix}$$

$$AZ=\begin{bmatrix}\{\beta_1S^0+(1-\varepsilon)\beta_1V^0-\omega_1\}I+\{\beta_2S^0+(1-\varepsilon)\beta_2V^0\}T\\ \phi I-\omega_2T\end{bmatrix}$$

$$G(X,Z)=\begin{bmatrix}\lambda S+(1-\varepsilon)\lambda V-\omega_1 I\\ \phi I-\omega_2 T\end{bmatrix}$$

Note that $\hat{G}(X,Z)=AZ-G(X,Z)$, this reduces to

$$\hat{G}(X,Z)=\begin{bmatrix}\beta_1\{(S^0-S)+(1-\varepsilon)(V^0-V)\}I+\beta_2\{(S^0-S)+(1-\varepsilon)(V^0-V)\}T\\ 0\end{bmatrix}.$$

$$\hat{G}(X,Z)=\begin{bmatrix}(S^0-S)(\beta_1I+\beta_2T)+(V^0-V)(1-\varepsilon)(\beta_1I+\beta_2T)\\ 0\end{bmatrix}.$$

$$\hat{G}(X,Z)=\begin{bmatrix}\hat{G}_1(X,Z)\\ \hat{G}_2(X,Z)\end{bmatrix}=\begin{bmatrix}(\beta_1I+\beta_2T)(S^0-S+(V^0-V)(1-\varepsilon))\\ 0\end{bmatrix}$$

Therefore, if $\hat{G}(X,Z)\geq 0$ then the DFE E^0 is globally stable, otherwise it is unstable.

Note that, $(1-\varepsilon)\geq 0$. The susceptible and vaccinated are bounded as, $V\leq V^0$ and $S\leq S^0$

Thus $\hat{G}_1(X,Z)$ and $\hat{G}_2(X,Z)$ are greater or equal to zero hence DFE, E^0 is globally asymptotically stable.

3.7 Local Stability and global stability of the Endemic Equilibrium point (EEP).

For system (1) - (4), when, it has a unique positive EEP, $E^*=(S^*,V^*,I^*,T^*)$ whenever $R_C>1$.

Consider a case where the drug is 100% efficient i.e $\varepsilon=1$, the systems of equations reduces to

$$\dot{S}=\Omega\pi-(\lambda+\theta+\mu)S+\gamma T+\tau I \quad (1),$$

$$\dot{V}=(1-\Omega)\pi-[\theta+\mu+(1-\varepsilon)\lambda]V+\rho I \quad (2),$$

$$\dot{I}=\lambda S+(1-\varepsilon)\lambda V-\omega_1 I \quad (3),$$

$$\dot{T}=\phi I-\omega_2 T \quad (4),$$

$$\lambda=\beta_1I+\beta_2T$$

$$R_C=\frac{\beta_1S^0}{\omega_1}+\frac{\phi\beta_2S^0}{\omega_1\omega_2}$$

Theorem 5

The EEP is globally asymptotically stable if $R_C>1$.

Proof

We propose the following Lyapunov function

$$K(S,I,T)=S-S^*-S^*\ln\frac{S}{S^*}+D_1\left(V-V^*-V^*\ln\frac{V}{V^*}\right)+D_2\left(I-I^*-I^*\ln\frac{I}{I^*}\right)$$
$$+D_3\left(T-T^*-T^*\ln\frac{T}{T^*}\right).$$

$$\frac{dK(S,V,I,T)}{dt} = \left(1-\frac{S^*}{S}\right)\frac{dS}{dt} + D_1\left(1-\frac{V^*}{V}\right)\frac{dV}{dt} + D_2\left(1-\frac{I^*}{I}\right)\frac{dI}{dt} + D_3\left(1-\frac{T^*}{T}\right)\frac{dI}{dt}$$

where D_1 $D_2$and D_3 are positive constants to be determined. This type of Lyapunov function has been mentioned in [2].

Which satisfies the conditions;

$$K(S^*,V^*,I^*,T^*) = 0 \quad \text{(i)},$$

$$K(S,V,I,T) > 0 \quad (ii)$$

Therefore $K(S,V,I,T)$ is positive definite.

For $\frac{dK(S,V,I,T)}{dt}$ to be negative definite, it must satisfies

$$\frac{dK(S^*,V^*,I^*,T^*)}{dt} = 0 \quad \text{(iv)}$$

$$\frac{dK(S,V,I,T)}{dt} < 0 \quad (v)$$

The endemic equilibrium point (EEP) for our system $E^* = (S^*,V^*,I^*,T^*)$ satisfies,

$\pi\Omega = (\lambda^* + \theta + \mu)S^* - \gamma T^* - \tau I^*,$
$(1-\Omega)\pi = [\theta + \mu + (1-\varepsilon)\lambda^*]V^* - \rho I^*.$
$\lambda^* S^* + (1-\varepsilon)\lambda^* V^* = \omega_1 I^*$
$\phi I^* = \omega_2 T^*$

$$\begin{aligned}\frac{dK(S,V,I,T)}{dt} = {} & \left(1-\frac{S^*}{S}\right)(\Omega\pi - (\lambda+\theta+\mu)S + \gamma T + \tau I) \\ & + D_1\left(1-\frac{V^*}{V}\right)((1-\Omega)\pi - [\theta+\mu+(1-\varepsilon)\lambda]V + \rho I) \\ & + D_2\left(1-\frac{I^*}{I}\right)(\lambda S + (1-\varepsilon)\lambda V - \omega_1 I) + D_3\left(1-\frac{T^*}{T}\right)(\phi I - \omega_2 T),\end{aligned}$$

Substituting for $\Omega\pi$ and $(1-\Omega)\pi$ at DFE to obtain,

$$\begin{aligned}\frac{dK(S,V,I,T)}{dt} = {} & \left(1-\frac{S^*}{S}\right)\left((\lambda^*+\theta+\mu)S^* - \gamma T^* - \tau I^* - (\lambda+\theta+\mu)S + \gamma T + \tau I\right) \\ & + D_1\left(1-\frac{V^*}{V}\right)([\theta+\mu+(1-\varepsilon)\lambda^*]V^* - \rho I^* - [\theta+\mu+(1-\varepsilon)\lambda]V + \rho I) \\ & + D_2\left(1-\frac{I^*}{I}\right)(\lambda S + (1-\varepsilon)\lambda V - \omega_1 I) + D_3\left(1-\frac{T^*}{T}\right)(\phi I - \omega_2 T),\end{aligned}$$

$$\begin{aligned}\frac{dK(S,V,I,T)}{dt} = {} & -(\mu+\theta)\frac{(S-S^*)^2}{S} - D_1(\mu+\theta)\frac{(V-V^*)^2}{V} - \gamma T^* - \tau I^* - D_1\rho I^* + D_3\phi I^* + \lambda^* S^*\{1+D_2\} \\ & + (1-\varepsilon)\lambda^* V^*\{D_1+D_2\} + I\{D_3\phi + D_1\rho - D_2\omega_1 + \tau + \beta_1 S^* + D_1(1-\varepsilon)\beta_1 V^*\} \\ & + T\{D_3 - \omega_2 + \gamma + \beta_2 S^* + D_1(1-\varepsilon)\beta_2 V^*\} + IS\{-\beta_1 + D_2\beta_1\} + TS\{-\beta_2 + D_2\beta_2\} \\ & + IV\{-D_1(1-\varepsilon)\beta_1 + D_2(1-\varepsilon)\beta_1\} \\ & + TV\{-D_1(1-\varepsilon)\beta_2 + D_2(1-\varepsilon)\beta_2\}\left(-\frac{S^*}{S}\right)\{\lambda^* S^* - \gamma T^* - \tau I^* + \gamma T + \tau I\} \\ & + D_1\left(-\frac{V^*}{V}\right)\{(1-\varepsilon)\lambda^* V^* - \rho I^* + \rho I\} + D_2\left(-\frac{I^*}{I}\right)\{\lambda S + (1-\varepsilon)\lambda V\} + D_3\left(-\frac{T^*}{T}\right)\phi I,\end{aligned}$$

The positive constants D_1, D_2 and D_3 are chosen such that coefficients of SI, TS, IV, TV and I are equal to zero

$$
\begin{aligned}
&-\beta_1 + D_2\beta_1 = 0, && \text{(IS)},\\
&-\beta_2 + D_2\beta_2 = 0, && \text{(TS)},\\
&-D_1(1-\varepsilon)\beta_1 + D_2(1-\varepsilon)\beta_1 = 0, && \text{(IV)},\\
&-D_1(1-\varepsilon)\beta_2 + D_2(1-\varepsilon)\beta_2 = 0, && \text{(TV)},\\
&D_3\phi + D_1\rho - D_2\omega_1 + \tau + \beta_1 S^* + D_1(1-\varepsilon)\beta_1 V^* = 0 && \text{(I)}.
\end{aligned}
$$

Solving the above equations, we obtain,

$$
\begin{aligned}
&D_1 = D_2 = 1,\\
&D_3\phi + \rho - \omega_1 + \tau + \beta_1 S^* + (1-\varepsilon)\beta_1 V^* = 0\\
&D_3 = \frac{\omega_1}{\phi} - \frac{\rho}{\phi} - \frac{\tau}{\phi} - \frac{\beta_1 S^* + (1-\varepsilon)\beta_1 V^*}{\phi}
\end{aligned}
$$

Substituting for the constants,

$$
\begin{aligned}
\frac{dK(S,V,I,T)}{dt} = &-(\mu+\theta)\frac{(S-S^*)^2}{S} - (\mu+\theta)\frac{(V-V^*)^2}{V} - \gamma T^* + \tau I\frac{T^*}{T} + \rho I\frac{T^*}{T} - 2\tau I^* - 2\rho I^*\\
&+\left\{-\frac{\beta_1 S^* + (1-\varepsilon)\beta_1 V^*}{\phi}\right\}\phi I^* + \lambda^* S^*\left(3 - \frac{S^*}{S}\right) + (1-\varepsilon)\lambda^* V^*\left(3 - \frac{V^*}{V}\right)\\
&+T\left\{\frac{\omega_1}{\phi} - \frac{\rho}{\phi} - \frac{\tau}{\phi} - \frac{\beta_1 S^* + (1-\varepsilon)\beta_1 V^*}{\phi} - \omega_2 + \gamma + \beta_2 S^* + (1-\varepsilon)\beta_2 V^*\right\}\\
&-\frac{S^*}{S}\{-\gamma T^* - \tau I^* + \gamma T + \tau I\} - \frac{V^*}{V}\{-\rho I^* + \rho I\} - \frac{I^*}{I}\{\lambda S + (1-\varepsilon)\lambda V\}\\
&+\left\{\frac{\omega_1}{\phi} - \frac{\beta_1 S^* + (1-\varepsilon)\beta_1 V^*}{\phi}\right\}\left(-\frac{T^*}{T}\right)\phi I
\end{aligned}
$$

Complete algebraic analysis of the above Lyapunov function to determine conditions necessary for local and global stability of the endemic equilibrium point will part of future research.

4 Analytical Results of the Model and their Biological Interpretation

4.1 Local and global stability of the Disease Free Equilibrium (DFE) point and Endemic Equilibrium Point (EEP).

When equilibrium point is locally stable, all the location near it tends to move towards it over time while equilibrium point is globally stable, all initial starting conditions lead to it over time.

4.2 The Equilibrium points and Thresholds

The control reproduction number R_C and basic reproduction number R_0 are given by

$$
\begin{aligned}
R_C &= \frac{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0}{\omega_1} + \frac{\phi\{\beta_2 S^0 + \beta_2(1-\varepsilon)V^0\}}{\omega_1\omega_2}\\
R_0 &= \frac{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0}{\omega_1 - \phi}
\end{aligned}
$$

The treatment threshold is determined when R_C is equated to one a solving for ϕ^C (critical treatment),

$$1 = \frac{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0}{\omega_1} + \frac{\phi^C\{\beta_2 S^0 + \beta_2(1-\varepsilon)V^0\}}{\omega_1\omega_2},$$

$$\phi^C = \frac{\omega_1\omega_2}{\{\beta_2 S^0 + \beta_2(1-\varepsilon)V^0\}} - \frac{\omega_2\{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0\}}{\{\beta_2 S^0 + \beta_2(1-\varepsilon)V^0\}}.$$

When actual treatment ϕ is more significant than critical treatment ϕ^C, it can ensure total eradication of pneumonia i.e

$$\phi > \phi^C.$$

Also, treatment with sufficient coverage can succeed in eliminating infection when R_c is below unity. Because R_c measures the intensity of the epidemic, treatment, by lowering R_c, can have significant public health impact even if it fails to eliminate infection in a specific population.

Following McLean and Blower, a measure of treatment impact based on the reproduction numbers can be defined as

$$(U) = 1 - \frac{R_C}{R_0},$$

$$(U) = 1 = \frac{\frac{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0}{\omega_1} + \frac{\phi\{\beta_2 S^0 + \beta_2(1-\varepsilon)V^0\}}{\omega_1\omega_2}}{\frac{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0}{\omega_1 - \phi}},$$

$$(U) = 1 - \frac{\omega_1 - \phi}{\omega_1} + \frac{\phi\beta_2(\omega_1 - \phi)}{\omega_1\omega_2\beta_1},$$

:

$$(U) = 1 - \frac{\theta + \mu + \tau + \rho + \delta_1}{\omega_1} + \frac{\phi\beta_2(\theta + \mu + \tau + \rho + \delta_1)}{\omega_1\omega_2\beta_1},$$

Thus, the population-level impact of treatment is always positive provided. This condition is likely to be satisfied with treatment with effective drugs.

Vaccination is a voluntary process, and it is not possible to vaccinate all individuals in the population. The herd immunity threshold is determined by

$$q_c = 1 - \frac{1}{R_0},$$

Where q_c is the critical vaccination threshold

$$q_c = 1 - \frac{1}{\frac{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0}{\omega_1 - \phi}},$$

$$q_c = 1 - \frac{\omega_1 - \phi}{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0}.$$

When actual vaccination $(I-\Omega)\pi$ is more celebrated than critical treatment $_cQC$ it can ensure total eradication of pneumonia i.e

$$(I-\Omega)\pi > q_c.$$

4.3 Sensitivity of effective control number R_C

It is important to investigate the sensitivity of R_C to changes in the rate at which infected (I) seek treatment ϕ, with respect to vaccination drug efficacy ε, rate at which treated (T) recover after treatment γ and rate at which infected (I) recover from natural immunity to susceptible and vaccinated classes (τ and ρ respectively). Determining partial derivatives of R_C concerning;

The

i. rate at which infected (I) seek treatment ϕ.

$$\frac{dR_C}{d\phi} = \frac{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0}{\omega_1} + \frac{\beta_2 S^0 + \beta_2(1-\varepsilon)V^0}{\omega_1\omega_2} > 0,$$

ii. The rate at which treated (T) recover after treatment γ.

$$\frac{dR_C}{d\gamma} = -\frac{\phi\{\beta_2 S^0 + \beta_2(1-\varepsilon)V^0\}}{\omega_1\omega_2^2} < 0,$$

iii. The rate at which infected (I) recover from natural immunity τ to susceptible class.

$$\frac{dR_C}{d\tau} = -\left(\frac{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0}{\omega_1^2} + \frac{\phi\{\beta_2 S^0 + \beta_2(1-\varepsilon)V^0\}}{\omega_2\omega_1^2}\right) < 0$$

iv. The rate at which infected (I) recover from natural immunity ρ to vaccinated class.

$$\frac{dR_C}{d\rho} = -\left(\frac{\beta_1 S^0 + \beta_1(1-\varepsilon)V^0}{\omega_1^2} + \frac{\phi\{\beta_2 S^0 + \beta_2(1-\varepsilon)V^0\}}{\omega_2\omega_1^2}\right) < 0.$$

v. With respect to vaccinated (V) drug efficacy ε.

$$\frac{dR_C}{d\,\varepsilon} = -\left(\frac{\beta_1 V^0}{\omega_1} + \frac{\phi\beta_2 V^0}{\omega_1\omega_2}\right) < 0$$

Clearly R_C is directly proportional to ϕ but inversely proportional to; ε, ρ, τ and γ.

5 Estimated numerical results

We will obtain estimated numerical results of the reproduction numbers and determine sensitivity analysis using graphically.

5.1The estimated numerical values

The estimated reproduction numbers is determined by substituting the estimated parameters in the table below in the analytical expressions ($R_C, R_1, R_2, R_3, R_4, R_5$ and R_0) as the proportion of the infectious population (infectious and treated), we obtain,

The parameters are summarized in the table below,

Parameters	π	τ	θ	ρ	ϕ	γ	μ	ε	β_1	β_2	Ω	δ_1	δ_2
Value	4000	0.0238	0.22	0.00367	0.22468	0.0476	0.0002	0.68	0.00022	0.000176	0.8	0.132	0.112
source	A	A	A	A	A	A	A	A	A	A	A	A	A

A- Assumed

$$R_C = 0.00132494, \quad R_1 = 0.00137905, R_2 = 0.00133301, R_3 = 0.00138779,$$
$$R_4 = 0.00143524, , R_5 = 0.00121276 \text{ and } R_0 = 0.0017943$$

We obtain the impact of treatment(U) based on the study (12) as,

$$(U) = 1 - \frac{R_C}{R_0} = 1 - \frac{0.00132494}{0.0017943} = 0.2616,$$

5.2 Sensitivity analysis

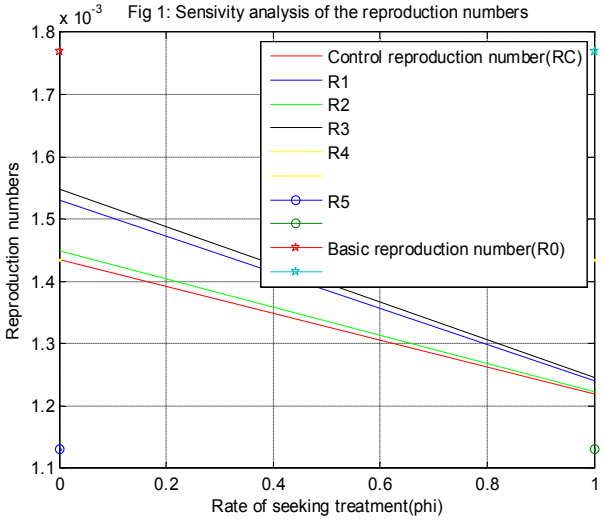


Fig. 1. The graph indicates that the reproduction number is inversely proportioned to the rate of seeking treatment (ϕ). Higher treatment rates hold great promise to lowering impact of pneumonia.

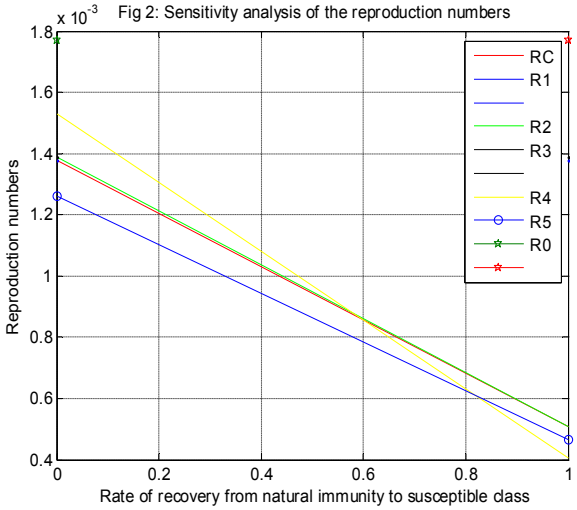


Fig. 2. The graph indicates that the reproduction number is inversely proportioning to the rate of recovery from natural immunity (τ). Higher recovery rates from innate resistance hold great promise to lowering impact of pneumonia

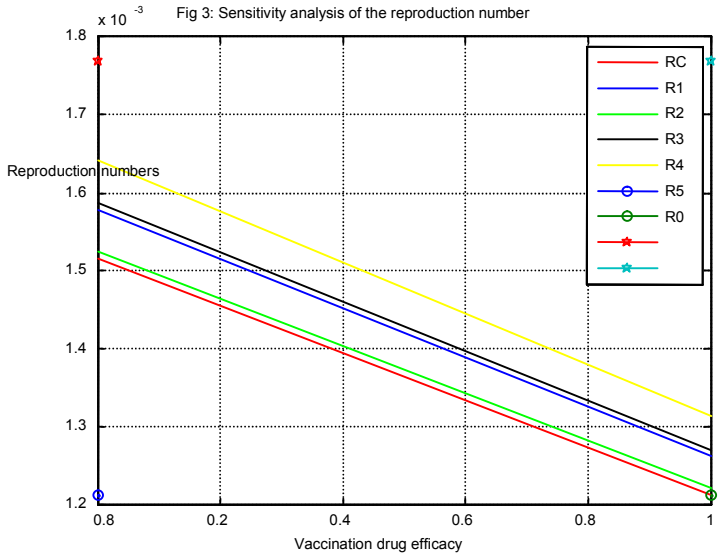


Fig. 3. The graph indicates that the reproduction number is inversely proportioned to the vaccination drug efficacy(ε) Higher drug efficacy hold great promise to lowering impact of pneumonia

6 Numerical simulation

The following initial conditions were used to carry out simulation,

$$S(0) = 20000, V(0) = 10, I(0) = 5 \text{ and } T(0) = 0$$

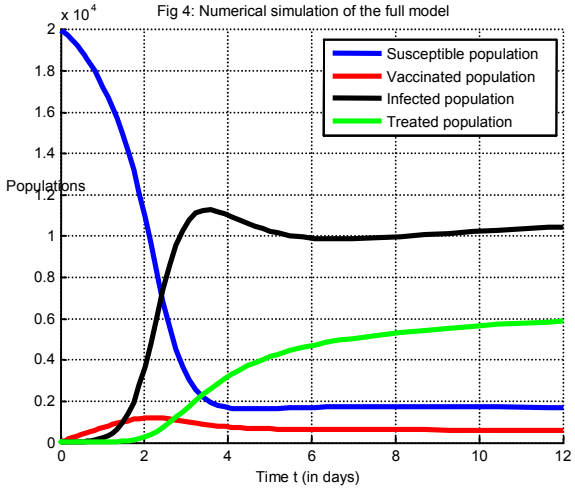


Fig. 4. Numerical simulation of the full model

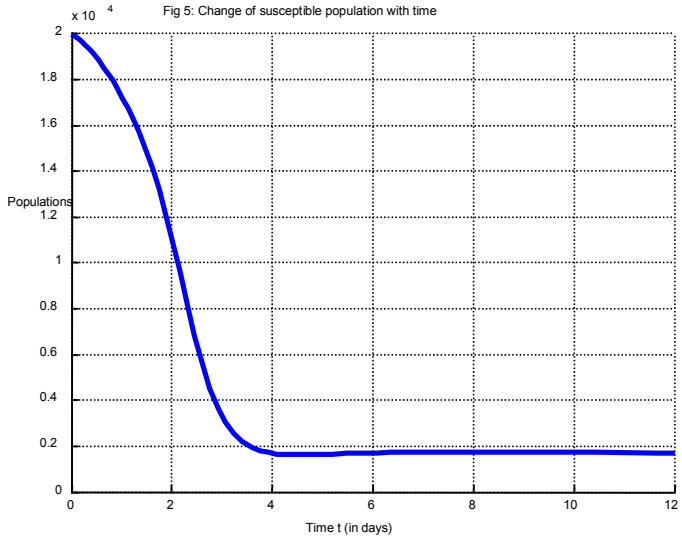


Fig. 5. Simulation of the susceptible population indicates that the community can decrease from 20,000 to 2000 in about four days and remain constant thereafter

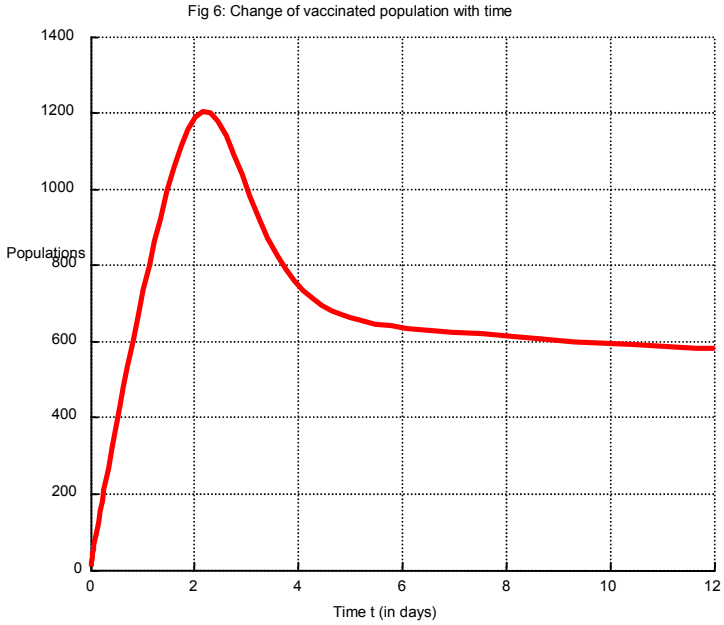


Fig. 6. Simulation of the vaccinated population indicates that the community can rapidly increase from 10 to 1200 in about two days, then decrease to about 600 in 6 days and remain constant thereafter

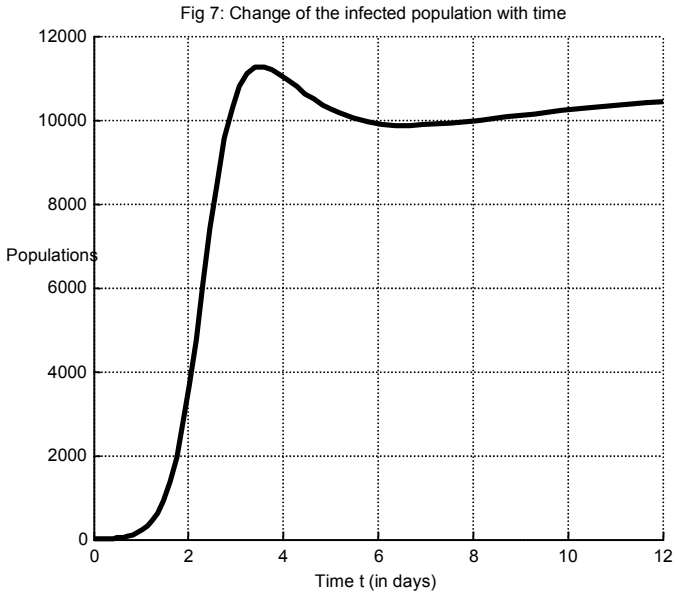


Fig. 7. Simulation of the infected population indicates that the community can rapidly increase from 5 to 11000 in about four days, then decrease to about 1000 in 6 days and increase gradually after that.

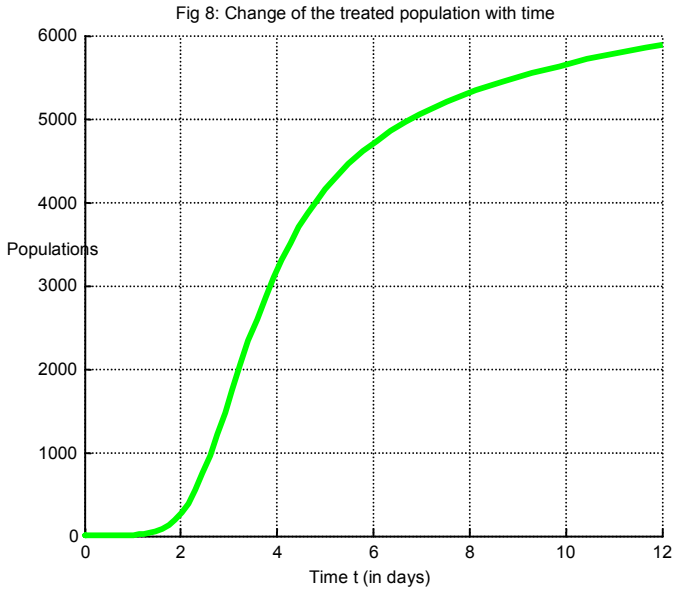


Fig. 8. Simulation of the treated population indicates that the community can rapidly increase from 0 to 6000 in about 12 days

7 Conclusion

Higher recovery rates after treatment (γ) with drugs, with higher vaccination efficacy (ε) and higher recovery rate from natural immunity (ν and ρ), would decrease the control reproduction number and the intensity of pneumonia endemic.

Competing Interest

Author has declared that no competing interests exist.

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APPENDIX

Table 1. Description of parameters and terms

Variables	Description
S(t)	The population of children less than five years of age susceptible to pneumonia.
V(t)	The population of children less than five years of age vaccinated against pneumonia.
I(t)	The population of children less than five years of age infected with pneumonia
T(t)	The population of children less than five years of age treated with pneumonia.
Parameters	**Description**
π	Recruitment rate (birth) of susceptible individuals.
5	Natural constant death rate.
β_1	The infection rates due to pneumonia in infected class.
β_2	The infection rates due to pneumonia in treated class.
τ	The rate at which infected children recover through natural immunity to susceptible class.
ρ	The rate at which infected children recover through natural immunity to vaccinated classes.
δ_1	Rate at which severely infected individual die due to Pneumonia.
δ_2	Rate at which severely treated individual die due to Pneumonia.
θ	The rate at which children under the age of five progress to the next age class per year.
γ	The rate at which treated children recover after treatment.
ϕ	The rate at which infected children seek treatment.
ε	The vaccination drug efficacy

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