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Fuzzy Risk Measure for Operational Risk

Assem Tharwat[1], Ramadan A. ZeinEldin[2], Hamiden Abd El-Wahed Khalifa[3] and Ahmed M. Saleim[4*]

[1]College of Business Administration, American University of the Emirates, Dubai, United Arab Emirates.
[2]Deanship of Scientific Research, King Abdul Aziz University, Kingdom of Saudi Arabia.
[3]Institute of Statistical Studies & Research, Cairo University, Egypt.
[4]Institute of Statistical Studies & Research, Cairo University, Egypt.

Authors' contributions

This work was carried out in collaboration between all authors. Author AMS designed the study, performed the statistical analysis and wrote the first draft of the manuscript. Authors AT and RAZE managed the analyses of the study. Author HAEWK managed the literature searches. All authors read and approved the final manuscript.

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Abstract

Operational risk is one of the most hazardous types of risk banks face. Banks must take caution and reserve capital to meet these risks. Value at Risk (VaR) and Expected Shortfall (ES) used to measure operational risk and estimate the required capital to meet it. Value at Risk is not sub-additive and measure risk at specific point in risk position, i.e. does not measure the risk in the tail, while Expected Shortfall examines only the left tails of the loss distribution. At the same time, to apply VaR or ES banks must have enough historical data. On the other hand, banks need an early warning indictor to monitor the movement of the capital needed to meet operational risk and take correction action in appropriate time. In this paper we will introduce new risk measure based on fuzzy numbers. The main advantage of the new risk measure is that the banks can use it as an early warning indictor to monitor the capital required to meet risk. At the same time can be used as an alternative to VaR but have more desirable properties. The application of the proposed risk measure shown that, the obtained results are more reliable and accurate at the same time the proposed risk measure have more desirable properties than VaR and ES.

**Corresponding author: E-mail:ahmedginny@hotmail.com;*

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1 Introduction

To find the capital to meet operational risk, under advanced measurement approach, we need to do two things. The first one is to estimate the expected operational loss. The most common and theoretical method to measure operational risk is loss distribution approach. The second thing, after find expected loss we need to map this loss to single number that represents the required capital to meet future loss. These done by using what we called risk measure. The most common risk measure is Value at Risk (VaR) and Expected Shortfall (ES).

VaR is calculated at a high specific level of confidence (99.9th percentile) and at a given period, at the same time, high severity/low frequency nature of operational risk data lead to limited data points at the tail of the loss distribution. This means that VaR does not estimate or predict loss due to extreme movements, i.e. VaR provides an estimate at particular points in the losses distribution, which mean it does not measure the risk in the tail. Indeed to that, VaR methodology based on normality assumption, so many authors criticized using VaR when the loss are not normally distributed, which is the case in operational risk, [1] and [2]. In 2014, the Basel Committee on Banking Supervision recommends the usage of Expected Shortfall. Since then the Expected Shortfall (ES) has been widely accepted as a risk measure but it has been criticized for issues relating to back-testing, [3] and [4].

In addition to criticism to VaR and ES and the necessity of availability of sufficient historical data, banks need to an indicator to monitor the amount of capital required to meet risk.

In this paper, fuzzy number will be used to introduce a new risk measure to estimate required capital to meet operational risk at the same time can used as an early warning indicator.

The remainder of this paper is organized as follows. Section 2 provides the background for the risk measure, the risk measure definition, literature review and survey, risk measure properties and popular risk measure. Section 3 gives some notationsabout the most important preliminary of fuzzy set which used in the paper. In section 4 we describe why we need a new risk measure. Section 5 devoted to present the solution methodology, the proposed risk measure, its properties and using it to determine the required capital to meet operational risk or as an early warning indicator. Application are presented and discussed in section 6. Conclusion was given in section 7.

2 Risk Measures

Risk measure is the most important step in risk management and play a crucial role for all parties involving in decisions making process (Regulators, Supervisors, Risk manager and top management, Public or private companies and Investors), [5,6] and [4].

a) Regulators use risk measure to provide rules to guarantee the stability of the system and determine the risk capital that bank will be required to hold as Basel II and Basel III Accords illustrated the importance of choosing a proper risk measure for regulatory purpose.
b) Supervisors need risk measure to ensure that the bank activities respect the legal framework.
c) Risk manager and top management, the decisions made by risk manager will be affected by risk measure and its accuracy, at the same time top management use risk measure for management purpose.
d) Public or private companies use risk measure to manage the wealth of their customers.
e) Investors use the risk measure to make investment decision since investor decision depend on the return as well as degree of risk.

Based on the aforementioned, having an accurate risk measure, to estimate the required capital, and having an early warning indictor, to monitor the amount of capital required to meet operational risk, is very important.

2.1 Risk measure definition

Risk measure provide a decision maker by a system to assign a single value to the set of random losses in order to make informed risk decisions, [7,6], and [3].

There are many definitions to risk measure. The most common and most formally definition define risk measure as a function ρwhich assign a single numerical value to the random loss, i.e. ρ map a distribution of a set of loss Ω to real numbers$\mathbb{R}$, that is $\rho : \Omega \rightarrow \mathbb{R}$, [8,6,3,4], and [9].

2.2 Literature review and survey

In 1930 Gramer introduce ruin theory. Ruin theory (sometimes called risk theory) uses mathematical models to describe an insurer's vulnerability to ruin/insolvency. In such models, key of interest are the probability of ruin. The classical model has two opposing cash flows: incoming cash premiums with constant rate of arrive and outgoing claims, which follow Poisson process. The central object of the model is to investigate the probability of ruin, [3] and https://en.wikipedia.org/wiki/Ruin_theory.

In 1952, Markowitz introduces modern portfolio theory, which uses the variance of the profit and loss as risk measure. However, it suffers from two drawbacks. It assume that the risk are random variables with finite variance and it suppose that the distributions are symmetric around the mean since the variance does not distinguish between positive and negative deviations from the mean, [10].

The Value at Risk (VaR) is widely used since introduced in 1990s as a single aggregate risk measure. VaR is the most widely risk measure in banks and can apply to all risk types. VaR captures the downside risk so it is used as basis for measuring risk-based capital, regulatory or economic capital, to ensure that the bank never become insolvent. VaR define the potential losses or maximum losses at a specific confidence level, [11] and [1].

In 1997 Wang, Young, and Panjer started first axiomatic approach to risk measure. They present four axioms to describe the behavior of market insurance prices (pure premium for an insurance risk) and propose an additional axiom for reducing compound risks. Thus, the insurance risk measures satisfy five axioms: Law invariance, Monotonicity, Comonotonic additivity, Continuity and Scale normalization, [12] and [6].

In 1998 Artzner, Delbaen, Eber, and Heath present a unified framework for construction, analysis, and implementation of measure of risk (market and non-market risks “they concentrate on the market risk”). They define the risk, based on the principle of bygones are bygones, as the variability of the future value of a position due uncertain events instead of define the risk as change in position value between two points. To achieve their goals they define a set called acceptable set$\mathcal{A}$ (set of acceptable net worths), which contain acceptable future net worth’s. The acceptable set are the basic object to be consider in order to decide accept or reject the risk, i.e. the risk measure can define by describing how close or far from acceptance position is. Thus, measurement of the risk of a position or portfolios currently held will be weather its future value belongs or does not belong to subset of acceptable risks. Then they define measure of risk of a position with an unacceptable future net worth (unacceptable risks) as the minimum capital required investing to make the future value of the new position or portfolios become acceptable. They also define a four axioms or properties (Translation invariance, Subadditivity, Positive homogeneity and Monotonicity) for risk measure and call risk measure, which satisfy these axioms coherent risk measure. They argue that to effectively regulate or measure risks, these axioms should hold for any risk measure, [13].

In 2013 Hede, Kou and Peng extended coherence risk measure by relaxes the subadditivity in the coherent risk measure and set two new axioms, Scenario-wise co-monotonic subadditivity and Empirical law invariance. Risk statistic satisfy the axioms translation invariance, positive homogeneity and monotonicity (in coherent risk measure) and the new two axioms Scenario-wise co-monotonic subadditivity and Empirical law invariance will be called natural risk statistic, [6] and [14].

In 2017 Eliza Khemissi [15] introduce additional axiom of measure of risk and review, which risk measure fulfill this additional axiom in order to enrich Arzner's and other axioms.

2.3 Risk measure properties

The desirable properties of risk measures have been established in a set of axioms. The axioms are stated in groups as follows:

Coherence risk measure, risk measure satisfies the following axioms called coherent risk measure:

- Translation invariance
- Sub-additive
- Positive homogeneity
- Monotonicity

Insurance risk measure, risk measure satisfies the following axioms called insurance risk measure:

- Monotonicity
- Co-monotonically additive
- Law invariance
- Continuity
- Scale normalization

Natural risk statistic, risk measure satisfies the following axioms called natural risk statistic:

- Translation invariance
- Positive homogeneity
- Monotonicity
- Scenario-wise co-monotonic sub-additive, Empirical law invariance

Convex risk measure, risk measure satisfies the following axioms called convex risk measure:

- Translation invariance
- Monotonicity
- Convexity

In addition to these set of axioms there are two significant properties (Elicit-ability and Robustness) should be considered, [3]:

2.3.1 The popular risk measure axioms

The popular properties (axioms) for risk measure was exhibiting and clarifying in the following. Let $\Omega = \{L_1, L_2, \dots, , L_m\}$be the set of random number represent the random losses associated with operational risk ($L_i$is the set of losses in the ith operational risk cell) and $\rho: L_i \to \mathbb{R}$be the risk measure.

Axioms 1: Translation invariance

Risk measure $\rho$is translation invariant if for all loss variables $L \in \Omega$and $a \in \mathbb{R}$it hold that$\rho(L + a) = \rho(L) + a$. Translation invariance, mean that if the loss increase (decrease) by the amount a the risk increase (decrease) by the same amount a. Translation invariance determines the necessary capital needed to compensate the risk, [3,6,16,13] and [4].

Axioms 2: Sub-additive

Risk measure ρ is sub-additive if for all loss variables, $L_1, L_2 \in \Omega$ it holds that $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2)$. Subadditivity mean that risk merger does not yield extra risk which is natural requirement, [15,3,6] and [13].

Axioms 3: Positive homogeneity

Risk measure ρ is positive homogeneous if for all variables $L \in \Omega$ and $h \in \mathbb{R}, h \geq 0$ it hold that $\rho(hL) = h\rho(L)$. Positive homogeneity means that, the risk measure at a specific position will double if the risk at these position doubles [9,3].

Axioms 4: Monotonicity

Risk measure ρ is monotonic if for all loss variables $L_1, L_2 \in \Omega$ with $L_1 \leq L_2$ then $\rho(L_1) \leq \rho(L_2)$. Monotonicity means if we have two portfolios one with payoffs "y" and the second with payoff "x" and first one dominates the second one then the portfolio with payoffs "y" must have less or equal risk than portfolio with payoffs "x". Monotonicity is the minimum requirement for a rational risk measure. Monotonicity ensures that the higher losses lead to higher risk measure, [15,17,3,6] and [4].

There is another property called co-monotonic additive, which considered as complementary to sub-additive property (Axioms 2), [3].

Axioms 5: Co-monotonically additive

Risk measure ρ is co-monotonic additive if for any co-monotonic random variable L_1 and L_2 it holding that $\rho(L_1 + L_2) = \rho(L_1) + \rho(L_2)$, [3].

A risk measure ρ called coherent risk measure if it satisfies axioms 1, 2, 3, and 4. A drawback of coherent risk measures is that they are not robust, [3] and [6].

Axioms 6: Law invariance

Risk measure ρ is called law invariant if the two random variables L_1 and L_2 have the same distribution then $\rho(L_1) = \rho(L_2)$, i.e. $P(L_1 \leq c) = P(L_2 \leq c) \Rightarrow \rho(L_1) = \rho(L_2) \forall c \in \mathbb{R}$. This axiom mean that the risk measure depends entirely on the distribution of the random variable associated to it, [9, 6] and [4].

Axioms 7: Continuity

For risk measure ρ and $d \in \mathbb{R}, d \geq 0$, ρ has satisfies $lim_{d \to 0+} \rho((L-d)^+) = \rho(L)$ and $lim_{d \to \infty} \rho(\min(L, d)) = \rho(L)$, in which, $(L-d)^+ = \max(L-d, 0)$, [6] and [12].

The first condition means that a small truncation in the loss variable results in small changes in capital. The second condition means that risk measure ρ can be estimated by approximating L by bounded variables, [12].

Axioms 8: Scale normalization

Risk measure ρ is scaled normalization if $\rho(1) = 1$, [6].

Any risk measure satisfies axioms 4, 5, 6, 7, and 8 will called insurance risk measures. Insurance risk measure is not sub-additive always, does not incorporate scenario analysis and does not enable to compare different distortion functions or different priors, [12] and [6].

Axioms 9: Scenario-wise co-monotonic sub-additive (or called co-monotonic sub-additive)

Risk measure ρ is scaled scenario-wise co-monotonic sub-additive if $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2)$for any L_1, L_2 that are scenario-wise co-monotonic. Scenario-wise co-monotonicity is counterpart of the notion of co-monotonic for two random variables. Scenario-wise co-monotonicity means that L_1 and L_2 in the same direction, [6] and [14].

Axioms 10: Empirical law invariance (or called Permutation invariance)

Empirical law invariance is counterpart of law invariance (axiom 6) for insurance risk measure. Empirical law invariance axiom means that if any two data $L_1$and L_2 has the same empirical distributions under each scenario then L_1 and L_2gives the same risk measurement, [14] and [6].

Natural risk statistic satisfies axioms 1, 3, 4, 9, and 10, [6] and [14].

Axiom 11: Convexity

The Axioms 2 and 3 (Sub-additive and Positive homogeneity) are relaxed to a single convexity axiom, equation (1). Convexity axiom relax sub-additive axiom since it ensure that diversification never increase risk measure, [4,6] and [9].

$$\rho(\lambda L_1 + (1-\lambda)L_2) \leq \lambda\rho(L_1) + (1-\lambda)\rho(L_2) \forall L_1, L_2 \in \Omega, \forall\lambda \in [0,1] \quad (1)$$

A risk measure ρcalled convex risk measure if it satisfies axiom 1 and axiom 4, and axiom 11, [9].

2.3.2 Elicit-ability and robustness

Elicit-ability

Elicit-ability is a criterion help in determine the optimal point forecasts. Elicit-ability definition is linked to the definition of scoring functions. Scoring function assign a numerical score to single valued point forecast depend on the predictive point and realization [3].

Robustness

A risk measure is said to be robust if it is quite insensitive to measurement errors. Robustness measure the effect of small change or deviation from the assumptions in data on the estimate of risk measure. If the risk measure not robustness then small change in data will produce a hug impact on risk measure estimation. To investigate robustness of risk measure, it is useful to consider the Wasserstein distance, [3,4] and [9].

2.4 Popular risk measure

2.4.1 Value at Risk (VaR)

Value at Risk is the maximum loss the bank may lose it with the current information at a specific level of confidence. Value at Risk (VaR) defines as a quantile of the distribution of aggregate losses. If X denote a loss random variable, $F_X = P(X \geq x)$be the cumulative distribution function of$X$and $\alpha \in [0,1]$ level of confidence. Value-at-Risk of X at the confidence level$\alpha\%$, denoted$VaR_\alpha(X)$, is the$\alpha\%$ percentile (or quantile) of the loss distribution of$X$. Formally, VaR has given by equation (2), [18,10,6] and [16]:

$$VaR_\alpha(X) = q_\alpha(X) = \inf\{x \in \mathbb{R}: F_X(x) \geq \alpha\} = \inf\{x \in \mathbb{R}: P(X \leq x) \geq \alpha\} \quad (2)$$

VaR can be used for estimate required capital to meet risk and provide a bias to compare the risk levels for different loss distribution but it suffer from drawbacks when using to modeling operational risk. VaR give us information about the minimum loss in the $100(1-\alpha)\%$ worst cases but does not provide any information about the size of the loss in the remainingα. Based on previous drawback, the VaR cannot use for comparison between different distributions of loss because it fail to distinguish between various tail behind theα. In addition, VaR dose not fulfill sub-additive properties, [16] and [9].

2.3.2 Expected Shortfall (ES)

The criticism directed to VaR triggered many attempts to modify VaR. These attempts lead to introduce new risk measure called Expected Shortfall (ES). Expected Shortfall also called Tail VaR, Super-quantile, Conditional VaR or Tail Conditional Expectation. ES instead of measuring the minimum loss incurred in an accepted percentage of worst cases, ES measure the expected loss continued in that portion of unfortunate probabilities. Unlike VaR, ES use the extreme losses in the tail of the distribution (left tail). There are different alternative formula for define ES, [15,16] and [9].

Susanne Emmer, et al. [3] defines Expected Shortfall (ES) as follows: let $X_i, i = (1,2,3, \dots m)$represent the loss in ith postion, $X = \sum_{i=1}^{m} X_i$ be the generic loss for one period, and then ES define by equation (3):

$$\begin{aligned} ES_\alpha(X) &= \frac{1}{1-\alpha}\int_\alpha^1 q_u(X)du \\ &= E[X|X \geq q_\alpha(X)] + (E[X|X \geq q_\alpha(X)] - q_\alpha(X))(\tfrac{P[X \geq q_\alpha(X)}{1-\alpha} - 1) \end{aligned} \quad (3)$$

Which reduced to $ES_\alpha(X) = E[X: X \geq q_\alpha(X)]$ if aggregate losses X have continuous distribution (if$P[X = q_\alpha(X)] = 0$).

3 Notations

In this section we give review to the important fuzzy notations used in this paper.

3.1 Fuzzy sets

Fuzzy set is a class of object with a continuum of grades of membership. Fuzzy set $\tilde{A}$can define as follow, let X be a universal set, then the fuzzy set $\tilde{A}$is given by: $\tilde{A} = \{(x, \mu_{\tilde{A}}(x): x \in X)\}$, where, $\mu_{\tilde{A}}(x)$ called the membership function and represent the grade of membership of x in$A$, [19]and [20].

3.2 $\alpha -$ cut of a fuzzy set (A_α)

Given a fuzzy set $\tilde{A}$ in universal set X The α $-$cut (or interval of confidence at level α or α cut level set) of the fuzzy set $\tilde{A}$ is a crisp set, A_α, that contains all the elements of X have membership values in $\tilde{A}$ greater than or equal to α, i.e $A_\alpha = \{x: \mu_{\tilde{A}}(x) \geq \alpha, x \in X, \alpha \in [0,1]\}$, [19]and [20].

3.3 Convex fuzzy set

A fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)\}$ in universal set is called convex if all A_α are convex set, i.e. for every element $x_1 \in A_\alpha$ and $x_2 \in A_\alpha$ and for every$\alpha \in [0,1]$,$\lambda x_1 + (1-\lambda)x_2 \in A_\alpha \forall \lambda \in [0,1]$, [19].

3.4 Fuzzy number

Fuzzy numbers allow us to build a mathematical model for linguistic variable or fuzzy environment. A fuzzy number is an extension of a regular number, it does not refer to one single or certain value but rather refer to a connected set of possible (imprecise or uncertain) values, where each possible value has its membership function (weight) between 0 and 1, [20] and [21].

The α cut operation can apply to the fuzzy number. Let$\tilde{A}$ be a fuzzy number then α cutA_α for$\tilde{A}$is given by the crisp interval$[a_1^{(\alpha)}, a_3^{(\alpha)}]$, and defined as$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}]$, [21].

3.5 A triangular fuzzy number

Among the different shapes of fuzzy number, triangular fuzzy number (TFN) is the most popular one. A triangular fuzzy number $\tilde{A}$ can be represented by a triplet $[a, b, c]$. This representation interpreted as membership function given by equation (4), [21] and [19]:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-\mathrm{a}}{\mathrm{b-c}} \mathrm{a} \le \mathrm{x} \le \mathrm{b} \\ \frac{\mathrm{c}-x}{\mathrm{c-b}} \mathrm{b} \le \mathrm{x} \le \mathrm{c} \end{cases} \quad (4)$$

3.6 Fuzzy arithmetic using α-cut

Palash Dutta, et al. [19] shown that α-cut method is general enough to deal with different type of fuzzy arithmetic including addition, subtraction, division, multiplication, extracting nth root, exponentiation, taking logarithm.

4 The Problem Description

Current risk measures for operational risk such as Value at Risk (VaR) and Expected Shortfall (ES) suffer from some problems as described above, moreover it require existence of adequate historical data to estimate the required capital to meet potential losses in future. In addition, both VaR and ES do not enable banks from monitor the changing in the required capital to meet operational risk during a year. That is to say, the bank reaction to meet future risks will be postpone to the end of financial period and based on historical data, which expose the bank to more risks. Therefore, the bank needs a new risk measure that goes beyond the flaws found in VaR and ES, at the same time banks must have an early warning indicator to monitor instantly the capital required to meet operational risk.

5 Solution Methodology

In the following fuzzy numbers will be used to present a new risk measure which can be used as an early warning indictor to instantaneously determine capital need to meet loss arise from operational risk at the same time can be used as an alternative to VaR and ES.

5.1 Fuzzyfication

Fuzzy numbers will be used to present a new risk measure (α_RM) enable banks in a continuous manner to monitor the amount of the expected capital to meet operational risk, using the currently (monthly/daily) operational loss without waiting to the end of financial period and can be used as an alternative to VaR and ES with more desirable properties. Triangular fuzzy number will be used to present such method stems from that the triplet $[abc]$ of the triangular fuzzy number can be viewed as the [minimum average maximum] expected loss, which is suitable to financial institutions.

To present the proposed method firstly we need to represent the operational loss in fuzzy triangular number, if there are only three operational losses, we can arrange it in ascending order and represent it by fuzzy triangular number as follows. Let $\mathrm{A} = (\mathrm{a_i}, \mathrm{a_{i+1}}, \mathrm{a_{i+2}})$ be any three operational loss such that$\mathrm{a_i} \ge \mathrm{a_{i+1}} \ge$

a_{i+2}. We can express the operational loss set A as triangular fuzzy numbers$\widetilde{A}_i = (a_i, a_{i+1}, a_{i+2})$ as given in Fig.(1).

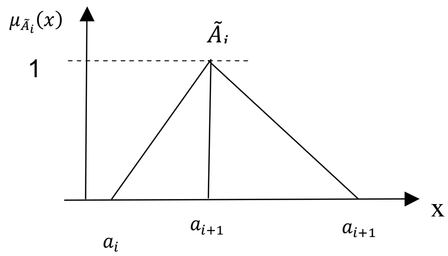

Fig. 1. Represent operational losses in fuzzy triangular number

With membership function of $\widetilde{A}_i$given by equation (7):

$$M_{\widetilde{A}_i}(x) = \begin{cases} \frac{x-a_i}{a_{i+1}-a_I} a_i \leq x \leq a_{i+1} \\ \frac{a_{i+2}-x}{a_{i+2}-a_{i+1}} a_{i+1} \leq x \leq a_{i+2} \end{cases} \tag{7}$$

Recalling the definition of α –cut, given a fuzzy set A in X and any real number$\alpha \in [0,1]$, then the α -cut of a set of A, denoted by A^α is the crisp set given by $A^\alpha = \{a \in A: \mu_a(x) \geq \alpha\}$, [16]. Thus α –cut can represent the confidence that the expected next loss fall in crisp set A^αwithαconfidence, [22]. Consequently, the capital required to cover the expected loss fall in the crisp set defined by equation (8):

$$A_i^\alpha = [(a_{i+1} - a_i)\alpha + a_i, a_{i+2} - (a_{i+2} - a_{i+1})\alpha] \tag{8}$$

α_RM for n operational loss can be summarized in following steps: let $\overline{A} = (\bar{a}_1, \bar{a}_2, \bar{a}_3, \dots, \bar{a}_n)$be a set of operational loss during specific period or part of current period.

1. Arrange $\overline{A} = (\bar{a}_1, \bar{a}_2, \bar{a}_3, \dots, \bar{a}_n)$ in ascending order. Let $A = (a_1, a_2, a_3, \dots, a_n)$ be the arranged operational loss.
2. Represent each three successive losses as triangular fuzzy numbers, thus we have $n - 2$ triangular fuzzy numbers as indict in Fig. (2), where $\widetilde{A}_1 = (a_1, a_2, a_3)$, $\widetilde{A}_2 = (a_2, a_3, a_4), \dots\dots$, $\widetilde{A}_i = (a_i, a_{i+1}, a_{i+2}), \dots\dots, \widetilde{A}_{n-2} = (a_{n-2}, a_{n-1}, a_n)$. The membership function for $\widetilde{A}_i$ is given by equation (9):

$$\mu_{\widetilde{A}_i}(x) = \begin{cases} \frac{x-a_i}{a_{i+1}-a_i} a_i \leq x \leq a_{i+1} \\ \frac{a_{i+2}-x}{a_{i+2}-a_{i+1}} a_{i+1} \leq x \leq a_{i+2} \end{cases} \tag{9}$$

3. α-cut of i fuzzy triangular number$\widetilde{A}_i$ given by equation (10):
4. $A_i^\alpha = [(a_{i+1} - a_i)\alpha + a_i, a_{i+2} - (a_{i+2} - a_{i+1})\alpha]$ (10)
5. The aggregate α-cut given by equation (11):

$$A = \sum_{i=1}^{n-2} \tilde{A}_i^{\alpha} = \tilde{A}_1^{\alpha} + \tilde{A}_2^{\alpha} + \cdots . + \tilde{A}_{n-2}^{\alpha}$$
$$= \left[(a_{n-1} - a_1)\alpha + \sum_{i=1}^{n-2} a_i \, , \sum_{i=3}^{n} a_i - (a_n - a_2)\alpha\right] \qquad (11)$$

with membership function given by equation (12):

$$\mu_A(x) = \begin{cases} \dfrac{x - \sum_{i=1}^{n-2} a_i}{\sum_{i=2}^{n-1} a_i - \sum_{i=1}^{n-2} a_i} = \dfrac{x - \sum_{i=1}^{n-2} a_i}{a_{n-1} - a_1} & \sum_{i=1}^{n-2} a_i \le x \le \sum_{i=2}^{n-1} a_i \\ \dfrac{\sum_{i=3}^{n} a_i - x}{\sum_{i=3}^{n} a_i - \sum_{i=2}^{n-1} a_i} = \dfrac{\sum_{i=3}^{n} a_i - x}{a_n - a_2} & \sum_{i=2}^{n-1} a_i \le x \le \sum_{i=3}^{n} a_i \end{cases} \qquad (12)$$

6. The required capital for operational risk falls in the crisp set given by the interval in equation (13).
   $C = [(a_{n-1} - a_1)\alpha + \sum_{i=1}^{n-2} a_i \, , \sum_{i=3}^{n} a_i - (a_n - a_2)\alpha]$ (13)
   Where, C is the required capital.

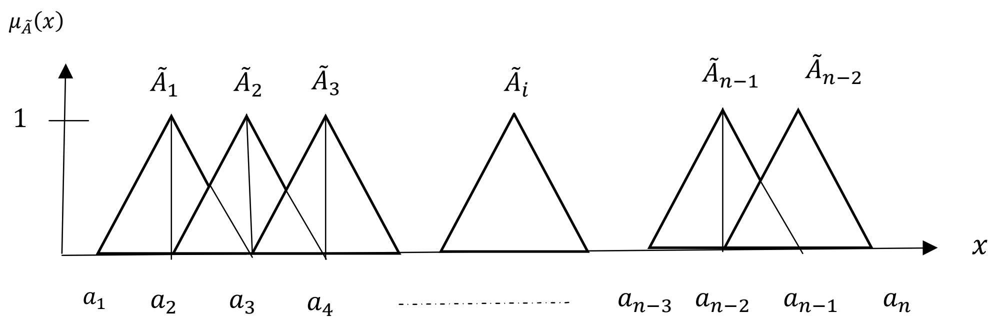

Fig. 2. Represent operational loss in TFN

Banks can use α_RM during the financial period to monitor the change in capital required to meet operational risk, increasing the required capital consistently and unnaturally or increase it above predetermined limit will give a pointer to bank to take a correction action.

5.2 Solution algorithm

To find capital required to meet operational risk, under the Advanced Measurement Approach (AMA), banks use the Loss Distribution Approach (LDA), which is the most common method used to measure operational risk. LDA is a frequency/severity model extensively used in many applications. It is a parametric technique use internal historical data and sometime enriched by external data to determine the frequency distribution and severity distribution of operational risk for each business line/event type (risk cell) at a specific time horizon. Then a suitable technique (e.g. Monte Carlo simulation or Panjer's recursive algorithms) used to combine the two distributions to obtain aggregate loss for the next period for each risk cell. Then bank can use the Value at Risk or Excepted Shortfall to determine the capital charge need to meet operational risk, [2].

In this section, we will present algorithm to use α_RM as an alternative to Value at Risk and Expected Shortfall to measure operational risk. Supposed that, there are a historical operational loss data, to find the required capital to meet operational loss next period, we follow the following steps:

1- Quantify the distribution of frequency and severity.
2- According to distribution of frequency, generate random number n represent the number of loss occur in the period.

3- According to distribution of severity, generate n losses represent the numbers of losses occur in one period.
4- Use α_RM to find the interval (lower bound and upper bound) of capital to cover these losses.
5- Repeat steps 2 and 4 m times.
6- The interval of the capital to meet the operational loss is the Quantile, at 99.9%, for lower and upper bound of the generated interval in previous step.

5. 3 α_RM properties as risk measure

By revising the axioms listed above, we find that some of them are redundant (counterpart to another) and others are not necessary to operational risk. Finally, the axioms to be proved if the proposed risk measure α _EWI satisfy it are axioms1 (Translation invariance), axioms2 (Sub-additive), axioms3 (Positive homogeneity), axioms4 (Monotonicity), axioms7 (Continuity), axioms8 (Scale normalization), axiom11 (Convexity), elicit-ability and robustness. In the following we prove that risk measure, α_RM satisfy the axioms 1, 2, 3, 4, 7, 8, and 11. We postpone verifying it is elicit-ability and robustness in further work.

1- Translation invariance (axioms 1): $\rho^{\alpha}(\tilde{L}+a)=\rho^{\alpha}(\tilde{L})+a$

The left hand side:

$$\rho^{\alpha}(\tilde{L}+a)=[((y_1+a)-(x_1+a))\alpha+(x_1+a),(z_1+a)-(\,(z_1+a)-(y_1+a))\alpha]$$
$$=[(y_1-x_1)\alpha+(x_1+a),(z_1+a)-(z_1-y_1)\alpha]$$

The right hand side:

$$\rho^{\alpha}(\tilde{L})+a=[(y_1-x_1)\alpha+x_1,z_1-(z_1-y_1)\alpha]+a$$
$$=[(y_1-x_1)\alpha+(x_1+a),(z_1+a)-(z_1-y_1)\alpha]$$

2- Sub-additive (axioms 2): if $L_1,L_2\in\Omega$ then $\rho^{\alpha}(\tilde{L}_1+\tilde{L}_2)\le\rho^{\alpha}(\tilde{L}_1)+\rho^{\alpha}(\tilde{L}_2)$

The right hand side:

$$\tilde{L}_1+\tilde{L}_2=\{(x_1+x_2),(y_1+y_2),(z_1+z_2)\}$$
$$\rho^{\alpha}(\tilde{L}_1+\tilde{L}_2)=[\big((y_1+y_2)-(x_1+x_2)\big)\alpha+(x_1+x_2),(z_1+z_2)-\big((z_1+z_2)-(y_1+y_2)\big)]$$

The left hand side:

$$\rho^{\alpha}(\tilde{L}_1)=[(y_1-x_1)\alpha+x_1,z_1-(z_1-y_1)\alpha]$$
$$\rho^{\alpha}(\tilde{L}_2)=[(y_2-x_2)\alpha+x_2,z_2-(z_2-y_2)\alpha]$$
$$\rho^{\alpha}(\tilde{L}_1)+\rho^{\alpha}(\tilde{L}_2)=[\big((y_1+y_2)-(x_1+x_2)\big)\alpha+(x_1+x_2),(z_1+z_2)-\big((z_1+z_2)-(y_1+y_2)\big)]$$

3- Positive homogeneity (axioms 3): $\rho^{\alpha}(h\tilde{L})=h\rho^{\alpha}(\tilde{L})$

$$h\tilde{L}=(hx_1,hy_1,hz_1)$$
$$\rho^{\alpha}(h\tilde{L})=[(hy_1-hx_1)\alpha+hx_1,hz_1-(hz_1-hy_1)\alpha]$$

The right hand side:

$$\rho^{\alpha}(\tilde{L})=[(y_1-x_1)\alpha+x_1,z_1-(z_1-y_1)\alpha]$$
$$h\rho^{\alpha}(\tilde{L})=h[(y_1-x_1)\alpha+x_1,z_1-(z_1-y_1)\alpha]$$
$$=[(hy_1-hx_1)\alpha+hx_1,hz_1-(hz_1-hy_1)\alpha]$$

4- Monotonicity (axioms 4): if $\tilde{L}_1\le\tilde{L}_2$then $\rho^{\alpha}(\tilde{L}_1)\le\rho^{\alpha}(\tilde{L}_2)$

$$\rho^{\alpha}(\tilde{L}_1)=[(y_1-x_1)\alpha+x_1,z_1-(z_1-y_1)\alpha]$$

$\rho^{\alpha}(\tilde{L}_2) = [(y_2 - x_2)\alpha + x_2, z_2 - (z_2 - y_2)\alpha]$

If $\tilde{L}_1 \leq \tilde{L}_2$ these means that $x_1 \leq x_2 \wedge y_1 \leq y_2 \wedge z_1 \leq z_2$
Since: $x_1 \leq x_2$ and $x_1 \leq x_2 \wedge y_1 \leq y_2$ then $(y_1 - x_1)\alpha \leq (y_2 - x_2)\alpha$ then $(y_1 - x_1)\alpha + x_1 \leq (y_2 - x_2)\alpha + x_2$

In the similar way $z_1 - (z_1 - y_1)\alpha \leq z_2 - (z_2 - y_2)\alpha$.

5- Scale normalization (axioms 7): $\rho^{\alpha}(1) = [1,1]$
$\rho^{\alpha}(\tilde{L}) = [(y_1 - x_1)\alpha + x_1, z_1 - (z_1 - y_1)\alpha]$
$\rho^{\alpha}(1) = [(1-1)\alpha + 1, 1 - (1-1)\alpha]$
$\rho^{\alpha}(1) = [1,1]$

6- Continuity (axioms 8):

We want to prove: $lim_{d\to 0+}\, \rho^{\alpha}((\tilde{L} - d)^{+}) = \rho^{\alpha}(\tilde{L})$

The right hand side:

$$\rho^{\alpha}(\tilde{L}) = [(y_1 - x_1)\alpha + x_1, z_1 - (z_1 - y_1)\alpha]$$

The left hand side:

$$\rho^{\alpha}((\tilde{L} - d)^{+}) = [((y_1 - d) - (x_1 - d))\alpha + (x_1 - d), (z_1 - d) - ((z_1 - d) - (y_1 - d))\alpha]$$

$$lim_{d\to 0+}\, \rho^{\alpha}((\tilde{L} - d)^{+}) = lim_{d\to 0+}\, [((y_1 - d) - (x_1 - d))\alpha + (x_1 - d), (z_1 - d) - ((z_1 - d) - (y_1 - d))\alpha]$$
$$= [(y_1 - x_1)\alpha + x_1, z_1 - (z_1 - y_1)\alpha] = \rho^{\alpha}(\tilde{L})$$

7- Convexity (axioms11): $\rho^{\alpha}(\lambda\tilde{L}_1 + (1-\lambda)\tilde{L}_2) \leq \lambda\rho^{\alpha}(\tilde{L}_1) + (1-\lambda)\rho^{\alpha}(\tilde{L}_2)$

From the definition of αcut and the fact that the, adding two triangular fuzzy numbers produce triangular fuzzy number. We can that the convexity condition holds.

6 Application

To test the accuracy of operational risk model, banks perform backtesting to these models. Backtesting used to analysis the differences between the prediction and the actual operational loss. Backtesting measure the accuracy of operational risk model by comparing the model output with the actual result during a specific period. There are four types of test can be used, clustering of violations, frequency of the violations, size of violations and size of over/under allocation of capital, [23]. Due to lack of data, we depend on the size of violations and size of over/under allocation of capital to validate our model.

To validate the α_RM the data obtained from [23] (fraud data and legal data) and from [24] (operational loss for bank) will be used.

6. 1 α_RM as an early warning indicator

To validate the α_RM as an early waning indicator, with extreme events, the fraud data obtained from [10] will be used. The fraud data collected from fraud events that took place between 1992 and 1996 and provided on monthly aggregate. The fraud data was summarized in Table (1).

In [23], they use Generalized Extreme Value distribution to find the amount of capital to cover fraud risk, the result summarized in Table (2):

Marcelo G.Cruz [23], returns the high amount of capital at 99% to the asymptotic characteristic of Generalized Extreme Value.

We apply α_RM to the fraud data in [7], for example in the first year; we have 10 triangular fuzzy number, $A_1 = (50{,}000\ \ 68{,}000\ 182{,}435)$, $A_2 = (68{,}000\ 182{,}435\ \ 220{,}357$, $\ldots A_{10} = (734{,}900\ \ 845{,}000\ \ 907{,}077)$. Then we find α-cut for each fuzzy triangular number$\widetilde{\mathrm{A}}_{\mathrm{i}}$ and the required capital fall in the interval of aggregate α-cut of$\widetilde{\mathrm{A}}_{\mathrm{i}}$. The results of applying α_RM are given in Table (3):

Table 1. Fraud data, [23]

month / year	1992	1993	1994	1995	1996
1	50,000.000	47,500.000	64,600.000	52,700.000	89,540.000
2	68,000.000	51,908.050	107,000.000	75,177.000	122,650.000
3	182,435.320	52,048.500	107,031.200	83,613.700	128,412.000
4	220,357.000	78,375.000	109,543.000	86,878.460	210,536.560
5	350,000.000	120,000.000	129,754.000	116,000.000	229,368.500
6	360,000.000	157,083.000	176,000.000	120,000.000	230,000.000
7	360,000.000	160,000.000	200,000.000	165,000.000	294,835.230
8	406,001.470	200,000.000	350,000.000	239,102.930	332,000.000
9	550,000.000	214,634.950	410,060.720	248,341.960	423,319.620
10	734,900.000	556,000.000	1,300,000.000	260,000.000	426,000.000
11	845,000.000	560,000.000	3,950,000.000	394,672.110	750,000.000
12	907,077.000	1,100,000.000	6,600,000.000	600,000.340	1,820,000.000
Sum	5,033,770.790	3,297,549.500	13,503,988.920	2,441,486.500	5,056,661.910
No. of fraud	586	454	485	658	798

Table 2.Capital required to cove fraud risk [23]

α	1992	1993	1994	1995	1996
99%	28,113,271	26,992,371	144,659,944	1,012,706	38,023,601
95%	5,656,815	5,522,432	22,135,683	444,486	7,311,518
90%	2,808,238	2,760,593	9,693,673	335,950	3,547,268
85%	1,851,669	1,827,292	5,911,090	292,046	2,302,561
80%	1,370,670	1,356,118	4,124,980	267,195	1,682,761

Table 3.The result of applying α_RMto fraud data in [7]

α	1992	1993	1994	1995	1996
0.999	[4,075,899 4,077,532]	[2,149,537 2,151,097]	[6,835,504 6,845,881]	[1,788,444 1,789,310]	[3,146,461 3,148,819]
0.950	[4,036,944 4,118,647]	[2,124,425 2,202,454]	[6,645,119 7,164,038]	[1,771,688 1,815,027]	[3,114,099 3,231,989]
0.001	[3,282,489 4,914,931]	[1,638,062 3,197,093]	[2,957,874 13,325,895]	[1,447,156 2,313,084]	[2,487,322 4,842,774]
0.050	[3,321,444 4,873,816]	[1,663,175 3,145,736]	[3,148,259 13,007,738]	[1,463,913 2,287,368]	[2,519,685 4,759,604]

The results obtained using GEV in [23] not realistic, [23] use 99%, 90%, 85%, and 80% percent to calculate the required capital to meet operational risk while the 99.9% is used in calculation. The capital estimated at

99% is much overestimated, at 95% is overestimated in 1994 and underestimated at 1995, and 85% or 80% is much underestimated and not associated with actual loss. On the other hand, the result obtained from α_RM in each year is very consistent to the actual operational loss, which means that we can depend on the proposed method to give bank general view about the amount of the required capital to meet operational loss, so we can use it as early warning indicator.

To test α_RM with operational loss follow distribution other than generalized extreme value distribution, the data obtained from [24] was used. Operational losses were collected during two years (2004, 2005) from South African retail bank. The operational loss (in Rand) given in Table (4).

Table 4.Operational losses summary, [24]

Sum	mean	Minimum	Maximum	Standard Deviation
31,586,348.00	277,073	1,000.00	9,669,000.00	1,032,814.09

Ja'NelEsterhuysen, et al. [24], uses the LDA (they use Poisson distribution for the frequency with mean equal 57 and Exponential distribution for severity distribution and generate 5000 yearly losses) they find that the VaR (required capital) at 99.9th is 13,384,748 Rand for one year.

We apply theα_RM to the same data given in [24]. The obtained results are given in Table 5.

Table 5. The result of applying α_RM to operational loss in [24]

α	The required capital
0.999	[10,931,279 11,006,514]
0.990	[10,931,279 11,006,514]
0.001	[8,271,364 15,787,306]
0.010	[8,295,569 15,743,800]

The required capital according to [24] is 13,384,748 Rand and the actual loss for year is 15,793,174 Rand. The obtained result by α_RM as shown in table [18] is consistent with the actual loss and with the result obtained by [24] which means that we can use the proposed method as an early warning indictor for operational risk.

6. 2 α_RM as an alternative to VaR and ES

To test the α_RM as a risk measure equivalent to VaR and ES but have more desirable properties, the fraud loss described in previous section and legal data from [23] will be was used to test the result are presents in following.

6.2.1 The estimated capital for fraud data in [23]

To validate the α_RM as a risk measure equivalent to VaR and ES but have more desirable properties, with extreme events, the fraud data obtained from [23] will be used

Marcelo G. Cruz [23], use Generalized Extreme Value to find the required capital for each years and he conclude that the high estimated capital due to high quantiles required for VaR as described in section 5.1. Therefore, we see that using GEV is no suitable and we fit the fraud data and find the required capital using VaR, ES and α_RM for each year using Poisson distribution for frequency and Exponential distribution for severity. The result of simulation with 10,000 runs given in Table 6.

Table (6) shown that, the results obtained from α_RM is consistent to the result obtained from VaR and ES and more reliable than using GEV which give a high estimated capital.

Table 6.The estimated capital using α_RM to fraud data in [23]

	1992	1993	1994	1995	1996
VaR at 99.9%	5,973,932	3,997,615	16,235,068	2,858,207	4,912,240
ES at 99.9%	5,852,970	3,891,325	15,822,683	2,814,362	4,828,310
α_QRM at 99.9%	**[4,720,118 5,270,209]**	**[3,134,743 3,516,099]**	**[12,774,334 14,284,808]**	**[2,263,260 2,515,480]**	**[3,918,151 4,268,070]**

6.2.2 The estimated capital for operational loss in [24]

To validate the α_RMthe operational loss in [24] was used. Ja'NelEsterhuysen, et. al. [24], use Poisson distribution for frequency and Exponential distribution for severity, the VaR (required capital) at 99.9th is 13 384 748 rand. We execute another simulation (10,000 runs) using the same distributions for frequency and the severity to find VaR, ES and α_RM. The result of simulation given in Table 7.

Table 7.The estimated capital using α_RM to fraud data in [24]

VaR	ES	α_RM
13,152,676	12,248,406	**[11,488,918 13,146,037]**

The actual (average) operational loss for one year is 15,793,174 Rand, using VaR the estimated required capital is13,152,676 Rand while using ES the required capital is 12,248,406 Rand. The results obtained from the proposed risk measure α_RM suggested to reserve capital in the range [11,488,918 13,146,037] to meet operational risk which is consistent to the result obtained from VaR and ES.

6.2.3 The estimated capital for legal loss in [23]

The legal loss consist of 75 loss, total loss is 32,974,449$, average of the loss is 739,299$, minimum loss 142,774$ and the maximum loss is 3,921,879$. Firstly, we use theα_RM to the 75 given loss to have an indication about the expected required capital to meet this loss, the result with different α given in Table (8).

Table 8. Estimated capital for legal loss in [23] usingα_RM

α	α_RM	
0.999	28,980,263	28,986,367.28
0.99	28,958,437	32,805,720.79
0.001	26,559,967	32,661,822.17
0.01	26,581,793	32,663,179.40

Secondly, we apply the steps given in section 5.2 to test α_RM as an alternative to VaR and ES, the number of runs is 10000. The required capital according to VaR is 46,590,199$ and the estimated capital using α_RM fall in the range [41,988,218 46,557,688].

Comparing between the total actual loss (32,974,449$), the result shown that we can use α_RM as early warning indicator, it's give us an idea about the amount of capital required to meet legal risk during the year as shown in table (8). This result enhance by the result obtained from the simulation (using α_RM as an alternative to VaR and ES) where the required capital fall in range [41,988,218 46,557,688].

7 Conclusion

The paper introduces new risk measure based on fuzzy (α_RM), these new risk measure present two contributions to improve risk measure for operational risk. The first contribution, banks can use α_RM as an early warning indicator to operational risk. Using α_RM atα = .001means that there are uncertainty in the environment surrounded the bank while using α = .999means more certainty. Using α_RM as early warning indicator will helps banks to monitor the required capital in a regular and progressive manner therefore banks can take preventive actions. The second contribution is that α_RM can be used as an alternative to VaR and ES but have more desirable properties.

The main advantage of the risk measure is it can be used to give indication to top management about the amount of required capital to meet operational risk without have a large historical database and it have desirable properties more than VaR and ES. The main disadvantage is that α_RM still depend on fitting the distribution for the frequency and severity, if the fitting is not correct the obtained result will be misleading.

Competing Interests

Authors have declared that no competing interests exist.

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