British Journal of Mathematics & Computer Science
19(3): 1-14, 2016; Article no.BJMCS.28927
ISSN: 2231-0851

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Coincidence Points & Common Fixed Points for Multiplicative Expansive Type Mappings

Sukh Raj Singh[1*], Manoj Ughade[2], R. D. Daheriya[1], Rashmi Jain[1] and Suraj Shrivastava[3]

[1]*Department of Mathematics, J H Government Post Graduate College, Betul, India.*
[2]*Department of Mathematics, Sarvepalli Radhakrishnan University, Bhopal, India.*
[3]*Department of Mathematics, Swami Vivekanand College of Science and Technology, Bhopal, India.*

Authors' contributions

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/BJMCS/2016/28927
Editor(s):
(1) Andrej V. Plotnikov, Department of Applied and Calculus Mathematics and CAD, Odessa State Academy of Civil Engineering and Architecture, Ukraine.
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Complete Peer review History: http://www.sciencedomain.org/review-history/16659

Original Research Article

Received: 13th August 2016
Accepted: 18th October 2016
Published: 25th October 2016

Abstract

In this paper, we prove some coincidence point and common fixed point results for various multiplicative expansive type mappings in the context of multiplicative metric spaces. We give some examples to demonstrate the validity of the results. Our results improve and supplement some recent results in the literature.

Keywords: Coincidence point; common fixed point; multiplicative metric space; expansive mapping.

2010 mathematics subject classification: 47H10, 54H25.

**Corresponding author: E-mail: srsinghssm@gmail.com;*

1 Introduction and Preliminaries

One of the simplest and most useful results in fixed point theory is the Banach-Caccioppoli Contraction mapping principle, a powerful tool in analysis for establishing existence and uniqueness of solution of problems in different fields. Over the years, this principle has been generalized in numerous directions in different spaces. These generalizations have been obtained either by extending the domain of the mapping or by considering a more general contractive condition on the mappings. In 1984, Wang et al. [1] introduced the concept of expansive mapping and established some fixed point theorems in complete metric spaces. In 1992, Daffer and Kaneko [2] proved some common fixed point theorems for two expansive mappings in complete metric spaces. For more results related to expansive mapping, see [3-4].

Bashirov et al. [5] studied the concept of multiplicative calculus and proved the fundamental theorem of multiplicative calculus. Florack and Assen [6] displayed the use of the concept of multiplicative calculus in biomedical image analysis. Bashirov et al. [7] exploit the efficiency of multiplicative calculus over the Newtonian calculus. They demonstrated that the multiplicative differential equations are more suitable than the ordinary differential equations in investigating some problems in various fields. Furthermore, Bashirov et al. [5] illustrated the usefulness of multiplicative calculus with some interesting applications. With the help of multiplicative absolute value function, they defined the multiplicative distance between two nonnegative real numbers as well as between two positive square matrices. This provides the basis for multiplicative metric spaces.

Definition 1.1 (see [8]) The multiplicative absolute value function $|x| : \mathbb{R}^+ \to \mathbb{R}^+$(where letter $\mathbb{R}^+$ denote the set of all nonnegative real numbers) is defined as

$$|x| = \begin{cases} x, & x \geq 1; \\ \frac{1}{x} & x < 1. \end{cases}$$

Definition 1.2 (see [5]) Let X be a nonempty set. A function $d: X \times X \to \mathbb{R}^+$ is said to be a multiplicative metric on X if for any $x, y, z \in X$, the following conditions hold:

(m1). $d(x, y) \geq 1$;
(m2). $d(x, y) = 1$ if and only if $x = y$;
(m3). $d(x, y) = d(y, x)$;
(m4). $d(x, y) \leq d(x, z). d(z, y)$.

The pair (X, d) is called a multiplicative metric space.

Example 1.3 (see [8]) Let $X = \mathbb{R}_n^+$ be the collection of all n-tuples of positive real numbers. Then $d(x, y) = \left|\frac{x_1}{y_1}\right| \left|\frac{x_2}{y_2}\right| \left|\frac{x_3}{y_3}\right| \dots \left|\frac{x_n}{y_n}\right|$ defines a multiplicative metric on X.

Example 1.4 Let $d: [0, +\infty) \times [0, +\infty) \to [1, +\infty)$ be defined as $d(x, y) = e^{|x-y|}$, where $x, y \in [0, +\infty)$. Then d is a multiplicative metric and $([0, +\infty), d)$ is a multiplicative metric space.

Remark 1.5 We note that the Example 1.3 is valid for positive real numbers and Example 1.4 is valid for all real numbers.

Example 1.6 (see [8]) Let (X, d) be a metric space. Define a mapping d_a on X by

$$d_a(x, y) = \begin{cases} 1, & x = y \\ a, & x \neq y \end{cases}$$

where $x, y \in X$ and $a > 1$. Then d_a is a multiplicative metric and (X, d_a) is known as the discrete multiplicative metric space.

Example 1.7 (see [9]) Let $X = C^*[a, b]$ be the collection of all real valued multiplicative continuous functions over $[a, b] \subseteq \mathbb{R}^+$. Then (X, d) is a multiplicative metric space with d defined by $d(f, g) = \sup_{x\in[a,b]} \left|\frac{f(x)}{g(x)}\right|$ for arbitrary $f, g \in X$.

Definition 1.8 (see [8]) Let (X, d) be a multiplicative metric space.

(1). A sequence $\{x_n\}$ in X is said to be multiplicative Cauchy sequence if for any $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) \le \epsilon$ for all $m, n > N$.
(2). A multiplicative metric space (X, d) is said to be complete if every Cauchy sequence $\{x_n\}$ in X is multiplicative convergent to a point $x \in X$.

A sequence $\{x_n\}$ in X is multiplicative Cauchy if and only if $d(x_n, x_m) \to 1$ as $n, m \to \infty$.

M. Sarwar and Badshah-e-Rome [10] discussed some unique fixed point theorems in multiplicative metric spaces. The established results carry some well known results from the literature to multiplicative metric spaces.

Definition 1.9 (see [11]) Let f be a mapping of a multiplicative metric space (X, d) into itself. Then f is said to be a multiplicative expansive mapping if there exists a constant $a > 1$ such that $d(fx, fy) \ge d^a(x, y)$ for all $x, y \in X$.

For more details of multiplicative metric space and related results, see [12-15].

Recently, Abodayeh et al. [16] and Agarwal et al. [17] studied the relationship between the multiplicative metric space and the standard metric space.

Definition 1.10 (see [18]) Let $f, g: X \to X$ be maps. A point $x \in X$ is called

(a) Fixed point of f if $fx = x$;
(b) Coincidence point of the pair (f, g) if $fx = gx$;
(c) Common fixed point of the pair (f, g) if $x = fx = gx$.

The sets of all fixed points of f, coincidence points of the pair (f, g), and all common fixed points of the pair (f, g) are denoted by $\mathcal{F}(f)$, $\mathcal{C}(f, g)$, and $\mathcal{F}(f, g)$, respectively.

2 Main Results

In this section, we will prove the existence of coincidence points and common fixed points of generalized multiplicative expansive mappings in the framework of multiplicative metric spaces.

We begin with a simple but a useful lemma.

Lemma 2.1 Let $\{x_n\}$ be a sequence in a multiplicative metric space X such that

$$d(x_n, x_{n+1}) \le d^\lambda(x_{n-1}, x_n) \tag{2.1}$$

where $\lambda \in [0, 1)$ and $n = 1, 2, \ldots$. Then $\{x_n\}$ is a Cauchy sequence in X.

Proof By induction, we have

$$d(x_n, x_{n+1}) \leq d^{\lambda^n}(x_0, x_1)$$

Let $m > n \geq 1$. From (m4), we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}).\, d(x_{n+1}, x_{n+2}) \ldots .\, d(x_{m-1}, x_m) \tag{2.2}$$

$$\leq d^{\lambda^n}(x_0, x_1).\, d^{\lambda^{n+1}}(x_0, x_1). \ldots .\, d^{\lambda^{m-1}}(x_0, x_1)$$

$$= d^{\lambda^n + \lambda^{n+2} + \lambda^{n+3} + \cdots + \lambda^{m-1}}(x_0, x_1)$$

$$\leq d^{\left(\frac{\lambda^n}{1-\lambda}\right)}(x_0, x_1)$$

Assume that $d(x_0, x_1) > 1$. Since $\lambda < 1$, by taking limit as $m, n \to +\infty$ in above inequality we obtain $lim_{n,m\to\infty}\, d(x_n, x_m) = 1$. Also, if $d(x_0, x_1) = 1$, then $d(x_n, x_m) = 1$ for all $m > n$. Hence $\{x_n\}$ is a multiplicative Cauchy sequence in X.

Now, we establish the following result of existence of common fixed points.

Theorem 2.2 Let (X, d) be a multiplicative metric space. Let $f, g: X \to X$ be surjective mappings satisfying

$$d(fx, gy) \geq \frac{d^a(x,y).d^b(x,fx).d^c(y,gy)}{[d(x,gy).d(y,fx)]^k} \tag{2.3}$$

for all $x, y \in X, x \neq y$, where $a, b, c, k \geq 0$ with $b < 1 + k, c < 1 + k, a > 1 + 2k$. Then f and g have a unique common fixed point in X.

Proof Let $x_0 \in X$ be an initial element and set $x_{2n} = fx_{2n+1}$, $x_{2n+1} = gx_{2n+2}$, $n = 0, 1, 2, \ldots$. Put $\lambda = \frac{1+k-b}{a+c-k}$. Since $a + b + c > 1 + 2k$, from (2.3), we have

$$d(fx_{2n+1}, gx_{2n+2}).\, d^k(x_{2n+1}, gx_{2n+2}).\, d^k(x_{2n+2}, fx_{2n+1})$$

$$\geq d^a(x_{2n+1}, x_{2n+2}).\, d^b(x_{2n+1}, fx_{2n+1}).\, d^c(x_{2n+2}, gx_{2n+2})$$

which implies

$$d(x_{2n}, x_{2n+1}).\, d^k(x_{2n+2}, x_{2n}) \geq d^{a+c}(x_{2n+1}, x_{2n+2}).\, d^b(x_{2n+1}, x_{2n})$$

Since $d(x_{2n+2}, x_{2n}) \leq d(x_{2n+2}, x_{2n+1}).\, d(x_{2n+1}, x_{2n})$, the last inequality gives us

$$d(x_{2n+1}, x_{2n+2}) \leq d^{\left(\frac{1+k-b}{a+c-k}\right)}(x_{2n}, x_{2n+1})$$

$$= d^{\lambda}(x_{2n}, x_{2n+1})$$

Similarly, we obtain

$$d(x_{2n+2}, x_{2n+3}) \leq d^{\lambda}(x_{2n+1}, x_{2n+2})$$

In general, we have

$$d(x_{n+1}, x_{n+2}) \leq d^{\lambda}(x_n, x_{n+1}) \tag{2.4}$$

and by Lemma 2.1, we deduce that $\{x_n\}$ is a multiplicative Cauchy sequence in X. Since (X, d) is complete, there exists $x^\star \in X$ such that $x_n \to x^\star$ as $n \to \infty$. Therefore $x_{2n+1} \to x^\star$ and $x_{2n+2} \to x^\star$ as $n \to \infty$. Since f and g are surjective mappings, there exist $u, v \in X$ such that

$$fu = x^\star \text{ and } gv = x^\star. \tag{2.5}$$

Applying (2.3) with $x = x_{2n+1}$ and $y = v$, we have

$$d(fx_{2n+1}, gv).\, d^k(x_{2n+1}, gv).\, d^k(v, fx_{2n+1})$$

$$\geq d^a(x_{2n+1}, v).\, d^b(x_{2n+1}, fx_{2n+1}).\, d^c(v, gv)$$

which implies that

$$d(x_{2n}, x^\star).\, d^k(x_{2n+1}, x^\star).\, d^k(v, x_{2n}) \tag{2.6}$$

$$\geq d^a(x_{2n+1}, v).\, d^b(x_{2n+1}, x_{2n}).\, d^c(v, x^\star)$$

Since $1 + 2k < a + b + c < a + 1 + k + c$, therefore $k < a + c$, on making limit as $n \to \infty$ in the above inequality and simplification leads to $d^{a+c-k}(v, x^\star) \leq 1$, which entails $d(v, x^\star) = 1$; that is, $v = x^\star$.

Again applying (2.3) with $x = u$ and $y = x_{2n+2}$, we have

$$d(fu, gx_{2n+2}).\, d^k(u, gx_{2n+2}).\, d^k(x_{2n+2}, fu)$$

$$\geq d^a(u, x_{2n+2}).\, d^b(u, fu).\, d^c(x_{2n+2}, gx_{2n+2})$$

this implies that

$$d(x^\star, x_{2n+1}).\, d^k(u, x_{2n+1}).\, d^k(x_{2n+1}, x^\star) \tag{2.7}$$

$$\geq d^a(u, x_{2n+2}).\, d^b(u, fu).\, d^c(x_{2n+2}, x_{2n+1})$$

Again since $1 + 2k < a + b + c < a + b + 1 + k$, thus $k < a + b$. On making limit as $n \to \infty$ in (2.7), we obtain that $d^{a+b-k}(u, x^\star) \leq 1$, which entails $d(u, x^\star) = 1$. Hence $u = x^\star$. The fact (2.5) along with $u = x^\star = v$ shows that $x^\star \in \mathcal{F}(f, g)$.

We are left to prove the uniqueness of the common fixed point $x^\star$; that is, $\mathcal{F}(f, g) = \{x^\star\}$. Let $x' \in \mathcal{F}(f, g)$ with $x^\star \neq x'$. By (2.3), we have

$$d(fx^\star, gx').\, d^k(x^\star, gx').\, d^k(x', fx^\star) \geq d^a(x^\star, x').\, d^b(x^\star, fx^\star).\, d^c(x', gx')$$

which on simplification leads to

$$d^{1+2k}(x^\star, x') \geq d^a(x^\star, x')$$

Since $1 + 2k < a$, we get $d^{a-1+2k}(x^\star, x') \leq 1$, which deduces $d(x^\star, x') = 1$. This contradicts (m2). Therefore $x^\star = x'$ and so $\mathcal{F}(f, g) = \{x^\star\}$. The proof is completed.

If in the above theorem 2.2 we take $b = c = k = 0$ and $f = g$, then we have the following corollary.

Corollary 2.3 (see [11], Theorem 2.1) Let (X, d) be a multiplicative metric space. Let $f: X \to X$ be a surjective mapping satisfying

$$d(fx, fy) \geq d^{a}(x, y) \tag{2.8}$$

for all $x, y \in X, x \neq y$, where $a > 1$. Then f has a unique fixed point in X.

If in the Theorem 2.2 we take $k = 0$ and $f = g$, then we have the following corollary.

Corollary 2.4 (see [10], Theorem 2.8) Let (X, d) be a multiplicative metric space. Let $f: X \to X$ be a surjective mapping satisfying

$$d(fx, fy) \geq d^{a}(x, y). d^{b}(x, fx). d^{c}(y, fy) \tag{2.9}$$

for all $x, y \in X, x \neq y$, where $a + b + c > 1, a > 1$ and $b < 1$. Then f has a unique fixed point in X.

If in the above corollary we take $b = c$, then we have the following corollary.

Corollary 2.5 (see [10], Corollary 2.8) Let (X, d) be a multiplicative metric space. Let $f: X \to X$ be a surjective mapping satisfying

$$d(fx, fy) \geq d^{a}(x, y). d^{b}(x, fx). d^{b}(y, fy) \tag{2.10}$$

for all $x, y \in X, x \neq y$, where $a + 2b > 1, a > 1$ and $b < 1$. Then f has a unique fixed point in X.

If we take $b = c = k = 0$ in Theorem 2.2, then we have the following corollary.

Corollary 2.6 Let (X, d) be a multiplicative metric space. Let $f, g: X \to X$ be surjective mappings satisfying

$$d(fx, gy) \geq d^{a}(x, y) \tag{2.11}$$

for all $x, y \in X, x \neq y$, where $a > 1$. Then f and g have a unique common fixed point in X.

If we take $k = 0$ in Theorem 2.2, then we have the following corollary.

Corollary 2.7 Let (X, d) be a multiplicative metric space. Let $f, g: X \to X$ be surjective mappings satisfying

$$d(fx, gy) \geq d^{a}(x, y). d^{b}(x, fx). d^{c}(y, gy) \tag{2.12}$$

for all $x, y \in X, x \neq y$, where $a, b, c \geq 0$ with $a + b + c > 1, b < 1, a > 1$. Then f and g have a unique common fixed point in X.

If we take $b = c$ in the Corollary 2.7 , then we have the following corollary.

Corollary 2.8 Let (X, d) be a multiplicative metric space. Let $f, g: X \to X$ be surjective mappings satisfying

$$d(fx, gy) \geq d^{a}(x, y). d^{b}(x, fx). d^{b}(y, gy) \tag{2.13}$$

for all $x, y \in X, x \neq y$, where $a + 2b > 1, a > 1$ and $b < 1$. Then f and g have a unique common fixed point in X.

Next, we prove the following result.

Theorem 2.9 Let $f, g: X \to X$ be two surjective mappings of a complete multiplicative metric space X. Suppose that f and g satisfying the following inequalities

$$d(fgx, gx) \geq \frac{d^a(gx,x)}{d^k(fgx,x)} \tag{2.14}$$

$$d(gfx, fx) \geq \frac{d^b(fx,x)}{d^k(gfx,x)} \tag{2.15}$$

for all $x \in X$ and some nonnegative real numbers a, b and k with $a > 1 + 2k$ and $b > 1 + 2k$. If f or g is continuous. Then f and g have a common fixed point.

Proof Let x_0 be an arbitrary point in X and set $x_{2n} = fx_{2n+1},\ x_{2n+1} = gx_{2n+2},\ n = 0, 1, 2, \ldots$. From (2.14), we have

$$d(fgx_{2n+2}, gx_{2n+2}).\, d^k(fgx_{2n+2}, x_{2n+2}) \geq d^a(gx_{2n+2}, x_{2n+2})$$

which implies

$$d(x_{2n}, x_{2n+1}).\, d^k(x_{2n}, x_{2n+2}) \geq d^a(x_{2n+1}, x_{2n+2})$$

Since $d(x_{2n}, x_{2n+2}) \leq\ d(x_{2n}, x_{2n+1}).\, d(x_{2n+1}, x_{2n+2})$, the last inequality gives us

$$d(x_{2n+1}, x_{2n+2}) \leq d^{\left(\frac{1+k}{a-k}\right)}(x_{2n}, x_{2n+2}) \tag{2.16}$$

On other hand, from (2.15) we have

$$d(gfx_{2n+1}, fx_{2n+1}).\, d^k(gfx_{2n+1}, x_{2n+1}) \geq d^b(fx_{2n+1}, x_{2n+1})$$

That is,

$$d(x_{2n-1}, x_{2n}).\, d^k(x_{2n-1}, x_{2n+`1}) \geq d^b(x_{2n}, x_{2n+1})$$

Since $d(x_{2n-1}, x_{2n+`1}) \leq\ d(x_{2n-1}, x_{2n}).\, d(x_{2n}, x_{2n+1})$, then from above inequality, we get

$$d(x_{2n}, x_{2n+1}) \leq d^{\left(\frac{1+k}{b-k}\right)}(x_{2n-1}, x_{2n}) \tag{2.17}$$

By combining (2.16) and (2.17), we obtain

$$d(x_n, x_{n+1}) \leq d^{\lambda}(x_{n-1}, x_n)$$

for all $n,$ where $\lambda = max\left\{\frac{1+k}{a-k}, \frac{1+k}{b-k}\right\}$. By Lemma 2.1, we deduce that $\{x_n\}$ is a multiplicative Cauchy sequence in X. Since X is complete, there exists $x^\star \in X$ such that $x_n \to x^\star$as $n \to \infty$. Therefore $x_{2n+1} \to x^\star$ and $x_{2n+2} \to x^\star$as $n \to \infty$. Without loss of generality, we may assume that f is continuous, then $fx_{2n+1} \to fx^\star$ as $n \to \infty$. But $fx_{2n+1} = x_{2n} \to x^\star$ as $n \to \infty$. Thus, we have $fx^\star = x^\star$. Since g is surjective, there exists $u \in X$ such that $gu = x^\star$. Now, applying (2.14), we have

$$d(fgu, gu) \geq \frac{d^a(gu,u)}{d^k(fgu,u)}$$

implies that

$$d^k(x^\star, u) \geq d^a(x^\star, u).$$

The last inequality gives us

$$d^{a-k}(x^{\star}, u) \leq 1 \tag{2.18}$$

Since $a-k > 1+k$, we infer from (2.18) that $d(x^{\star}, u) = 1$ and consequently, $x^{\star} = u$. Hence $x^{\star} \in \mathcal{F}(f, g)$.

By taking $b = a$ in above Theorem 2.9, we have the following result.

Corollary 2.10 Let $f, g: X \to X$ be two surjective mappings of a complete multiplicative metric space X. Suppose that f and g satisfying the following inequalities

$$d(fgx, gx) \geq \frac{d^{a}(gx,x)}{d^{k}(fgx,x)} \tag{2.19}$$

$$d(gfx, fx) \geq \frac{d^{a}(fx,x)}{d^{k}(gfx,x)} \tag{2.20}$$

for all $x \in X$ and some nonnegative real numbers a and k with $a > 1+2k$. If f or g is continuous. Then f and g have a common fixed point.

If we take $f = g$ in Corollary 2.10 we get the following corollary.

Corollary 2.11 Let $f: X \to X$ be a surjective mapping of a complete multiplicative metric space (X, d). Suppose that f satisfies the following inequality:

$$d(f^2x, fx) \geq \frac{d^{a}(fx,x)}{d^{k}(f^2x,x)} \tag{2.21}$$

for all $x \in X$ and some nonnegative real numbers a and k with $a > 1+2k$. If f is continuous. Then f has a fixed point.

If in the above Corollary 2.11, we take $k = 0$, then we have the following corollary.

Corollary 2.12 (see [10], Theorem 2.9) Let $f: X \to X$ be a surjective mapping of a complete multiplicative metric space X. Suppose that f satisfies the following inequality:

$$d(f^2x, fx) \geq d^{a}(fx, x) \tag{2.22}$$

for all $x \in X$, where $a > 1$. If f is continuous. Then f has a fixed point.

Next, we give a result of existence of coincidence point.

Theorem 2.13 Let (X, d) be a multiplicative metric space. Let $f, g: X \to X$ be mappings satisfying

$$d(fx, fy) \geq d^{a}(gx, gy).\, d^{b}(gx, fx).\, d^{c}(gy, fy) \tag{2.23}$$

for all $x, y \in X$, where $a, b, c \geq 0$ with $a+b+c > 1$. Suppose the following hypotheses:

1) $b < 1$ or $c < 1$;
2) $gX \subseteq fX$;
3) fX is a complete subspace of X.

Then f and g have a coincidence point.

Proof Let $x_0 \in X$. Since $gX \subseteq fX$, we choose $x_1 \in X$ such that $fx_1 = gx_0$. Again we can choose $x_2 \in X$ such that $fx_2 = gx_1$. Continuing in the same way, we construct a sequence $\{x_n\}$ in X such that $fx_{n+1} =$

$gx_n, \forall\, n \in \mathbb{N}$. If $gx_{m-1} = gx_m$ for $m \in \mathbb{N}$, then $fx_m = gx_m$. Thus x_m is a coincidence point of f and g. Now, assume that $gx_{n-1} \neq gx_n$ for all n. Set $\lambda = \max\left\{\frac{1-b}{a+c}, \frac{1-c}{a+b}\right\}$. We have the following two cases:

Case 1. Suppose $b < 1$. Applying (2.23), we have

$$\begin{aligned} d(gx_{n-1}, gx_n) &= d(fx_n, fx_{n+1}) \\ &\geq d^a(gx_n, gx_{n+1}).\, d^b(gx_n, fx_n).\, d^c(gx_{n+1}, fx_{n+1}) \\ &= d^a(gx_n, gx_{n+1}).\, d^b(gx_n, gx_{n-1}).\, d^c(gx_{n+1}, gx_n) \end{aligned}$$

The last inequality gives us

$$d(gx_n, gx_{n+1}) \leq d^{\left(\frac{1-b}{a+c}\right)}(gx_{n-1}, gx_n) \tag{2.24}$$

Case 2. Suppose $c < 1$. From (2.23) again, we have

$$\begin{aligned} d(gx_n, gx_{n-1}) &= d(fx_{n+1}, fx_n) \\ &\geq d^a(gx_{n+1}, gx_n).\, d^b(gx_{n+1}, fx_{n+1}).\, d^c(gx_n, fx_n) \\ &= d^a(gx_{n+1}, gx_n).\, d^b(gx_{n+1}, gx_n).\, d^c(gx_n, gx_{n-1}) \end{aligned}$$

This implies that

$$d(gx_n, gx_{n+1}) \leq d^{\left(\frac{1-c}{a+b}\right)}(gx_{n-1}, gx_n) \tag{2.25}$$

Combining (2.24) and (2.25), we get

$$d(gx_n, gx_{n+1}) \leq d^{\lambda}(gx_{n-1}, gx_n) \tag{2.26}$$

By induction on n, we obtain

$$d(gx_n, gx_{n+1}) \leq d^{\lambda^n}(gx_0, gx_1) \tag{2.27}$$

Thus for $m > n, n, m \in \mathbb{N}$ and since $\lambda < 1$, we have

$$\begin{aligned} d(gx_n, gx_m) &\leq d(gx_n, gx_{n+1}).\, d(gx_{n+1}, gx_{n+2}) \ldots .\, d(gx_{m-1}, gx_m) \\ &\leq d^{\lambda^n}(gx_0, gx_1).\, d^{\lambda^{n+1}}(gx_0, gx_1). \ldots .\, d^{\lambda^{m-1}}(gx_0, gx_1) \\ &= d^{\lambda^n + \lambda^{n+2} + \lambda^{n+3} + \cdots + \lambda^{m-1}}(gx_0, gx_1) \\ &\leq d^{\left(\frac{\lambda^n}{1-\lambda}\right)}(gx_0, gx_1) \end{aligned} \tag{2.28}$$

Assume that $d(gx_0, gx_1) > 1$. By taking limit as $m, n \to +\infty$ in inequality (2.28), we have $lim_{n,m\to\infty}\, d(gx_n, gx_m) = 1$. Also, if $(gx_0, gx_1) = 1$, then $d(gx_n, gx_m) = 1$ for all $m > n$. Therefore $\{fx_n\} = \{gx_n\}$ is a multiplicative Cauchy sequence in fX. Since fX is a complete subspace of X, there is $x^\star \in X$ such that $\{fx_n\}$ converges $fx^\star$ as $n \to \infty$. Hence $\{gx_n\}$ converges to $fx^\star$ as $n \to \infty$. Since $a + b + c > 1$, we have a, b and c are not all 0. So we have the following cases.

Case 1 If $a \neq 0$, then from (2.23), we have

$$d(fx_n, fx^\star) \geq d^a(gx_n, gx^\star).\, d^b(gx_n, fx_n).\, d^c(gx^\star, fx^\star)$$

$$\geq d^a(gx_n, gx^\star)$$

On letting limit as $n \to \infty$ in above inequality, we have

$$\lim_{n\to\infty} d(gx_n, gx^\star) \leq \lim_{n\to\infty} d^{\frac{1}{a}}(fx_n, fx^\star) = 1$$

Thus $\lim_{n\to\infty} d(gx_n, gx^\star) = 1$ and consequently $gx_n \to gx^\star$ as $n \to \infty$. By uniqueness of limit, we have $fx^\star = gx^\star$. Therefore f and g have a coincidence point.

Case 2 If $b \neq 0$, from (2.23), we have

$$d(fx^\star, fx_n) \geq d^a(gx_n, gx^\star).\, d^b(gx^\star, fx^\star).\, d^c(gx_n, fx_n)$$

$$\geq d^b(gx^\star, fx^\star)$$

On letting limit as $n \to \infty$ in above inequality, we have

$$d(fx^\star, gx^\star) \leq \lim_{n\to\infty} d^{\frac{1}{b}}(fx^\star, fx_n) = 1$$

Hence $d(fx^\star, gx^\star) = 1$ and consequently, $fx^\star = gx^\star$.

Case 3 If $c \neq 0$, again from (2.23), we have

$$d(fx_n, fx^\star) \geq d^a(gx_n, gx^\star).\, d^b(gx_n, fx_n).\, d^c(gx^\star, fx^\star)$$

$$\geq d^c(gx^\star, fx^\star)$$

On letting limit as $n \to \infty$ in above inequality, we have

$$d(fx^\star, gx^\star) \leq \lim_{n\to\infty} d^{\frac{1}{c}}(fx_n, fx^\star) = 1$$

Hence $d(fx^\star, gx^\star) = 1$ and consequently, $fx^\star = gx^\star$. Therefore f and g have a coincidence point.

Setting $c = 0$ in Theorem 2.13, we can obtain the following result.

Corollary 2.14 Let (X, d) be a multiplicative metric space. Let $f, g: X \to X$ be mappings satisfying

$$d(fx, fy) \geq d^a(gx, gy).\, d^b(gx, fx) \tag{2.29}$$

for all $x, y \in X$, where $a, b \geq 0$ with $a + b > 1$. Suppose the following hypotheses:

1) $b < 1$;
2) $gX \subseteq fX$;
3) fX is a complete subspace of X.

Then f and g have a coincidence point.

Setting $b = c = 0$ in Theorem 2.13, we can obtain the following corollary.

Corollary 2.15 Let (X, d) be a multiplicative metric space. Let $f, g: X \to X$ be mappings satisfying

$$d(fx, fy) \geq d^a(gx, gy) \tag{2.30}$$

for all $x, y \in X$, where $a > 1$. Suppose the following hypotheses:

1) $gX \subseteq fX$;
2) fX is a complete subspace of X.

Then f and g have a coincidence point.

Setting $g = I$ (Identity map) in Theorem 2.13, we have the following corollary.

Corollary 2.16 Let (X, d) be a complete multiplicative metric space. Let $f: X \to X$ be mapping satisfying

$$d(fx, fy) \geq d^a(x, y). d^b(x, fx). d^c(y, fy) \tag{2.31}$$

for all $x, y \in X$, where $a, b, c \geq 0$ with $a + b + c > 1$. Suppose $b < 1$ or $c < 1$. Then f has a fixed point.

Setting $b = c = 0$ in Corollary 2.16, we can obtain the following corollary.

Corollary 2.17 Let (X, d) be a complete multiplicative metric space. Let $f: X \to X$ be mapping satisfying

$$d(fx, fy) \geq d^a(x, y) \tag{2.32}$$

for all $x, y \in X$, where $a > 1$. Then f has a fixed point.

Setting $c = 0$ in Corollary 2.16, we can obtain the following corollary.

Corollary 2.18 Let (X, d) be a complete multiplicative metric space. Let $f: X \to X$ be mapping satisfying

$$d(fx, fy) \geq d^a(x, y). d^b(x, fx) \tag{2.33}$$

for all $x, y \in X$, where $a, b \geq 0$ with $a + b > 1$. Suppose $b < 1$. Then f has a fixed point.

3 Examples

In this section, we give some examples in support of results.

Example 3.1 Let $X = \mathbb{R}_+$ and define a mapping $d: X \times X \to \mathbb{R}$ by $d(x, y) = \left|\frac{x}{y}\right|$ for all $x, y \in X$. Then (X, d) is a complete multiplicative metric space. Define $f: X \to X$ by $fx = x^2$ for all $x \in X$. Then f is a surjection and continuous on X. Note that

$$\begin{aligned} d(f^2x, fx). d^{\frac{1}{4}}(f^2x, x) &= \left|\frac{x^4}{x^2}\right|. \left|\frac{x^4}{x}\right|^{\frac{1}{4}} \\ &= |x|^{\frac{11}{4}} \\ &\geq \left|\frac{x^2}{x}\right|^{\frac{5}{2}} \\ &= d^{\frac{5}{2}}(fx, x) \end{aligned}$$

where $k = \frac{1}{4}$ and $a = \frac{5}{2}$. Clearly $\frac{5}{2} = a > 1 + 2k = \frac{3}{2}$. Therefore, Corollary 2.12 is applicable to f and $x^\star = 0 \in X$ is a fixed point of f.

Example 3.2 Let $X = [0, \infty)$ be the usual metric space and define a mapping $d : X \times X \to \mathbb{R}$ by $d(x, y) = e^{|x-y|}$ for all $x, y \in X$. Then (X, d) is a complete multiplicative metric space. Define $f: X \to X$ by $fx = 2x$ for all $x \in X$. Then f is a surjection on X. Note that

$$\begin{aligned} d(f^2x, fx). d(f^2x, x) &= e^{|4x-2x|}. e^{|4x-x|} \\ &= e^{|2x|}. e^{|3x|} \\ &\geq \left(e^{|x|}\right)^5 = \left(e^{|2x-x|}\right)^5 \\ &= d^5(fx, x) \end{aligned}$$

where $k = 1$ and $a = 5$. Clearly $5 = a > 1 + 2k = 3$. Then (2.22) is satisfied. Therefore, Corollary 2.12 is applicable to f and $x^\star = 0 \in \mathcal{F}(f)$. Also Theorem 2.1 of [10] is applicable f.

Finally, we present an example to support the validity of Corollary 2.15.

Example 3.3 Let $X = [0, \infty)$ be the usual metric space and define a mapping $d : X \times X \to \mathbb{R}$ by $d(x, y) = e^{|x-y|}$ for all $x, y \in X$. Define $f, g: X \to X$ by $fx = \frac{x}{4}$ and $gx = \frac{x}{16}$ for all $x \in X$. Then f and g are surjection on X. Then $gX \subseteq fX$ and fX is complete. Note that

$$\begin{aligned} d(fx, fy) &= e^{|fx-fy|} \\ &= e^{\left|\frac{x}{4}-\frac{y}{4}\right|} \\ &= e^{4\left|\frac{x}{16}-\frac{y}{16}\right|} \\ &\geq e^{3|gx-gy|} \\ &= d^3(gx, gy) \end{aligned}$$

for all $x, y \in X$, where $a = 3 > 1$. Thus, Corollary 2.15 is applicable to f and g. Here $0 \in \mathcal{C}(f, g)$.

4 Conclusion

In this article, we established some coincidence point and common fixed point theorems for various multiplicative expansive-type mappings in the context of multiplicative metric spaces. The presented theorems extend, generalize and improve many existing results in the literature. Our results may be the motivation to other authors for extending and improving these results to be suitable tools for their applications.

Acknowledgements

The authors express deep gratitude to the referee for his/her valuable comments and suggestions.

Competing Interests

Authors have declared that no competing interests exist.

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