British Journal of Mathematics & Computer Science
19(3): 1-7, 2016; Article no.BJMCS.28879
ISSN: 2231-0851

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Square-Normal Operator

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/BJMCS/2016/28879
Editor(s):
(1) Fangfei Li, Department of Mathematics, East China University of Science and Technology, Shanghai, China.
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Complete Peer review History: http://www.sciencedomain.org/review-history/16665

Original Research Article

Received: 11th August 2016
Accepted: 6th October 2016
Published: 25th October 2016

Abstract

In this paper we define a new class of operators in Hilbert space called square-normal operator and we give an example to show that the square-normal operator is not normal operator. We also consider the conditions on any operator to be a square-normal operator. Then we give a condition in order to get a normal operator from a square-normal operator.

Keywords: Normal operators; numerical range of operators; self-adjoint operators; square-normal operators.

1 Introduction

In this paper A, A_1 , A_2, N, M and E represent continuous linear operators on Hilbert space H. If A is an operator on H, then we denote W(A) for the numerical range of A, A* denotes the adjoint of A, σ (A) denotes the spectrum of A, A is said to be normal if AA*=A*A, it is called self-adjoint if A=A* and unitary if AA*=A*A = I where I is the identity operator. The real part of A is denoted by $Re\,A = \frac{A+A^*}{2}$ and

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imaginary part of A is denoted by $A = \frac{A-A^*}{2i}$. Some of the standard textbooks on bounded operator theory are ([1], [2] and [3].

For a bounded linear operator A on H, the numerical range W(A) is the image of the unit sphere of H under the quadratic form $x \rightarrow < A_x, x >$ associated with the operator. More precisely,

$$W(A) = \{< A_x, x >: x \in H, ||x|| = 1\}.$$

The following special notations will be used throughout this paper: B= $A^2 (A^*)^2$ and C = $(A^*)^2 A^2$, where B and C are nonnegative definite.

In [4] author's results provide canonical matrices of linear operators A: U → U such that A^2 is a normal operator and U is a unitary or Euclidean space since changes of the basis transform the matrix of A by unitary or, respectively, orthogonal similarity. In this paper we will study the square-normal operator and its properties.

2 Square-Normal Operator and Some Properties

We now define the following operator:

Definition 1. *An operator A is said to be square-normal operator if* $A^2(A^*)^2 = (A^*)^2A^2$.

Proposition 1. *If A is a normal operator then A is a square-normal operator.*

Proof. *A* is normal operator then

$$A^2(A^*)^2 = AAA^*A^* = AA^*AA^* = A^*AA^*A = A^*A^*AA = (A^*)^2A^2.$$

So *A* is a square-normal operator.

The converse is not true. We give an example of a square-normal operator which is not normal:

Example 1.

$$A = \begin{matrix} i & 0 \\ i & -i \end{matrix} \quad , \quad A^* = \begin{matrix} -i & -i \\ 0 & i \end{matrix}$$

Since $A^2 = \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix}$ and $(A^*)^2 = \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix}$,

$A^2(A^*)^2 = \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix}$ and $(A^*)^2A^2 = \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix}$. *So A is a square-normal operator.*

But $AA^* = \begin{matrix} 1 & 1 \\ 1 & 2 \end{matrix}$ and $A^*A = \begin{matrix} 2 & -1 \\ -1 & 1 \end{matrix}$. *So A is not normal.*

Proposition 2. *A is a square-normal operator if and only if* A^2 *is normal.*

Proof : Let *A* be a square-normal operator, so

$$A^2(A^*)^2 = (A^*)^2A^2$$

$$\Leftarrow\Rightarrow A^2(A^2)^* = (A^2)^*A^2$$

$$\Leftarrow\Rightarrow A^2 \text{ is normal}$$

Theorem 1. *(Fuglede-Putnam* [5, 6]*). If A, N and M are bounded operators such that M and N are normal, then*

$$AN = MA \Rightarrow AN^* = M^*A,$$

and if N and M are unbounded, then " = " is replaced by " ⊂ " in the last displayed equation.

Now we state Embry's Theorem and some of its corollaries which we will use to prove our results. This Theorem and corollaries are stated in [7]:

Theorem 2. *If N and M are commuting normal operators and AN = MA, where 0 is not in the numerical range of A, then N=M.*

Corollary 1. *If A is an operator such that either $\sigma(A) \cap \sigma(-A) = \phi$ or $\mathbf{0} \notin W(A)$ and $A\,E = -E\,A$, where either A or E is normal, then E=0.*

Corollary 2. *If A^2 is normal and $0 \notin W(A)$, then A is normal.*

3 The Main Results

Theorem 3. *The following are equivalent*

i. *A is a square-normal.*

ii. *Each of B and C commute with Re A^2.*

iii *B commutes with Re A^2 and C commutes with Im A^2.*

Proof: (i) ⟺ (ii)

$$\begin{aligned} \mathrm{B}\,(\mathrm{Re}\,\mathrm{A}^2) &= \mathrm{B}\left(\frac{A^2+(A^*)^2}{2}\right) = \frac{B\,A^2+B(A^*)^2}{2} \\ &= \frac{A^2(A^*)^2A^2 + A^2(A^*)^2(A^*)^2}{2} \\ &= \frac{A^2A^2(A^*)^2 + (A^*)^2A^2(A^*)^2}{2} \\ &= \frac{A^2\,B + (A^*)^2\,B}{2} = \left(\frac{A^2+(A^*)^2}{2}\right) B = (Re\,A^2)\,B \end{aligned}$$

$$\begin{aligned} C\,(Re\,A^2) &= C\left(\frac{A^2+(A^*)^2}{2}\right) = \frac{C\,A^2+C(A^*)^2}{2} \\ &= \frac{(A^*)^2A^2A^2 + (A^*)^2A^2(A^*)^2}{2} \\ &= \frac{A^2(A^*)^2A^2 + (A^*)^2(A^*)^2A^2}{2} \\ &= \frac{A^2\,C + (A^*)^2\,C}{2} = \left(\frac{A^2+(A^*)^2}{2}\right) C = (Re\,A^2)\,C \end{aligned}$$

Conversely, let $H = Re\,A^2$ and $K = Im\,A^2$.

$$B\,A^2 = A^2(A^*)^2\,A^2 = A^2\,C \tag{1}$$

also we have

$$(A^*)^2B = (A^*)^2\, A^2(A^*)^2 = C\,(A^*)^2. \tag{2}$$

If

$$B\,H = H\,B \qquad and \qquad C\,H = H\,C$$

Then

$$\text{B}\,\left(\frac{A^2 + (A^*)^2}{2}\right) = \left(\frac{A^2 + (A^*)^2}{2}\right) B$$

$$B\,A^2 + \;B\,(A^*)^2 = A^2\,B + \,(A^*)^2\,B.$$

This equivalent to

$$B\,(A^*)^2 - \,(A^*)^2\,B = \,A^2\,B - B\,A^2\,.$$

Hence by (1) and (2) we have:

$$(\,B - C\,)\,(A^*)^2 = \,A^2\,(\,B - C\,) \tag{3}$$

And similarly

$$(\,B - C\,)\,A^2 = \,(A^*)^2\,(\,B - C\,) \tag{4}$$

Multiplying (3) on the left by $(A^*)^2$ and multiplying (4) on the right by$(A^*)^2$, we get:

$$(\,B - C\,)A^2\,(A^*)^2 = \,(A^*)^2\,A^2\,(\,B - C\,)$$

$$(\,B - C\,)B = C\,(\,B - C\,) \tag{5}$$

Multiplying (3) on the right by A^2 and multiplying (4) on the left by A^2, we get:

$$(\,B - C\,)(A^*)^2\,A^2 = \,A^2\,(A^*)^2\,(\,B - C\,)$$

$$(\,B - C\,)C = B\,(\,B - C\,) \tag{6}$$

Subtract (6) from (5), will give

$$(\,B - C\,)^2 = \,-\,(B - C)^2$$

$(B - C)^2 = 0$ implies $B - C = 0$ since B and C are self-adjoint operators, and so is $B - C$.

So A is square-normal operator.

(ii) $\Longrightarrow$ (iii)

$B(ReA^2) = (ReA^2)\,B$ is clear.

From (ii) $\Longrightarrow$ (i) we have A is a square-normal operator so we have

$$A^2\,C = C\,A^2 \qquad \& \qquad (A^*)^2\,C = C\,(A^*)^2 \tag{7}$$

So by (7) we have

$$C\ (Im\ A^2) = \mathrm{C}\left(\frac{A^2-(A^*)^2}{2i}\right) = \frac{C\ A^2 - C(A^*)^2}{2i}$$

$$= \frac{A^2\ C\ - (A^*)^2\ C}{2i}$$

$$= \left(\frac{A^2-(A^*)^2}{2i}\right) C = (\ Im\ A^2)\ C$$

(iii) ⟹ (i)

$$K\ C = C\ K$$

$$\left(\frac{A^2-(A^*)^2}{2i}\right) C = \mathrm{C}\left(\frac{A^2-(A^*)^2}{2i}\right)$$

$$(A^2-(A^*)^2)C = C\ (\ A^2-(A^*)^2)$$

$$A^2\ C\ -\ (A^*)^2\ C\ = C\ A^2 - C\ (A^*)^2$$

$$B\ A^2 -\ C\ A^2 =\ (A^*)^2\ C\ -\ (A^*)^2\ B$$

$$(\ B-C\ )\ A^2 =\ -(A^*)^2\ (\ B-C\ ) \qquad (8)$$

Since because of *HB* = *BH* then (3) is satisfied. Indeed, if

$$B\ H = H\ B$$

then

$$\mathrm{B}\left(\frac{A^2+(A^*)^2}{2}\right) = \left(\frac{A^2+(A^*)^2}{2}\right) B$$

$$B\ A^2 +\ B\ (A^*)^2 = A^2\ B +\ (A^*)^2\ B.$$

This equivalent to

$$B\ (A^*)^2 -\ (A^*)^2\ B =\ A^2\ B - B\ A^2\ .$$

Hence by (1) and (2), (3) is satisfied.

Multiplying (8) on the left by A^2 and multiplying (3) on the right by A^2, we get:

$$(\ B-C\ )C =\ -B\ (\ B-C\ )$$

$$B\ C -\ C^2 = B\ C -\ B^2$$

We get $C^2 = B^2$ and so $B = C$ ([8], p. 262).

Hence A is a square-normal operator.

Theorem 4: Let S and T be two square-normal operators. For any bounded linear operator A if

$$AS^2 = T^2A$$

Then

$$A\,(S^*)^2 = (T^*)^2\,A$$

Proof. Since S and T are square-normal operators, so by Proposition 2 we see that S^2 and T^2 are normal operators. So by Fuglede-Putnam Theorem 1 we see

$$A\,(S^*)^2 = (T^*)^2\,A$$

Theorem 5: Let S and T be two square-normal operators which commute, and let A be any bounded operator for which $0 \notin W(A)$. If $AS^2 = T^2A$ then $S^2 = T^2$.

Proof: Since S and T are square-normal operators, so by Proposition 2 we see that S^2 and T^2 are normal, so Embry's Theorem 2 is applicable, resulting in $S^2 = T^2$.

Theorem 6: If A is a square-normal operator and $0 \notin W(A)$ then A is normal.

Proof: If A is square-normal operator then by Proposition 2 A^2 is normal. So by Corollary 2 we see that A is normal.

Theorem 7: If B and C commute and $0 \notin W(A2)$ then A is square-normal operator.

Proof: Notice that

$$A^2\,(A^*)^2\,A^2 = A^2\,(A^*)^2\,A^2$$

$$A^2\,C = B\,A^2$$

Since B and C are normal even self-adjoint which commute and $0 \notin W(A^2)$. So Embry's Theorem 2 is applicable, resulting in $B = C$. So A is square-normal operator.

Theorem 8: Let A be any bounded operator. for any bounded operator E for which either $0 \notin W(E)$ or $\sigma(E) \cap \sigma(-E) = \emptyset$ if

$$EB + BE = 0 \tag{9}$$

and

$$EC + CE = 0 \tag{10}$$

then A is square-normal, where $B = A^2(A^*)^2$ and $C = (A^*)^2A^2$.

Proof: Subtracting (10) from (9), we see

$$EB - EC + BE - CE = 0$$

$$E(B - C) = -(B - C)E.$$

Since $0 \notin W(E)$ or $\sigma(E) \cap \sigma(-E) = \emptyset$ and $B - C$ is normal (even self-adjoint). So by Corollary 1 we see

$$B - C = 0 \Longrightarrow B = C.$$

So A is a square-normal operator.

4 Conclusion and Future Work

In this paper has presented new class of operator called square-normal operator and its properties. We see that every normal operator is a square-normal operator. As for future work, we plan to generalize this class of operators to n-normal operator and satisfy the same properties. We ask a question: Is the same results still true with the n-normal operator?

Competing Interests

Author has declared that no competing interests exist.

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