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Existence and Nonexistence of Positive Solutions for a System of Higher-Order Differential Equations with Integral Boundary Conditions

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, we investigate the existence and nonexistence of positive solutions for a system of nonlinear higher-order ordinary differential equations with Riemann-Stieltjes integral boundary conditions which contain some positive constants.

Keywords: Higher-order differential equations; integral boundary conditions; positive solutions; existence; nonexistence.

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1 Introduction

We consider the system of nonlinear higher-order ordinary differential equations

$$(S)\qquad \begin{cases} u^{(n)}(t) + a(t)f(v(t)) = 0, & t \in (0,T), \\ v^{(m)}(t) + b(t)g(u(t)) = 0, & t \in (0,T), \end{cases}$$

with the integral boundary conditions

$$(BC)\qquad \begin{cases} u(0) = \int_0^T u(s)\,dH_1(s) + a_0, \ \ u'(0) = \cdots = u^{(n-2)}(0) = 0, \ \ u(T) = \int_0^T u(s)\,dH_2(s), \\ v(0) = \int_0^T v(s)\,dK_1(s) + b_0, \ \ v'(0) = \cdots = v^{(m-2)}(0) = 0, \ \ v(T) = \int_0^T v(s)\,dK_2(s), \end{cases}$$

where $T > 0$, $n, m \in \mathbb{N}$, $n, m \geq 2$, the integrals from the boundary conditions (BC) are Riemann-Stieltjes integrals, $H_i, K_i : [0,T] \to \mathbb{R}$, $i = 1,2$ are functions of bounded variation, and a_0 and b_0 are positive constants. In the case $n = 2$ or $m = 2$ the boundary conditions above are of the form $u(0) = \int_0^T u(s)\,dH_1(s) + a_0$, $u(T) = \int_0^T u(s)\,dH_2(s)$, or $v(0) = \int_0^T v(s)\,dK_1(s) + b_0$, $v(T) = \int_0^T v(s)\,dK_2(s)$, respectively, that is, without conditions on the derivatives of u and v at point 0. The Riemann-Stieltjes integral boundary conditions (BC) cover the Riemann integral boundary conditions (when the functions H_1, H_2, K_1, K_2 are continuously differentiable functions), the multi-point boundary conditions (when the functions H_1, H_2, K_1, K_2 are step functions), and combinations between them.

By using the Schauder fixed point theorem and some properties of the associated Green's functions, we prove the existence of positive solutions of problem $(S)-(BC)$ for a_0, b_0 sufficiently small. By a positive solution of $(S)-(BC)$ we mean a pair of functions $(u,v) \in C^n([0,T];\mathbb{R}_+) \times C^m([0,T];\mathbb{R}_+)$ satisfying (S) and (BC) with $u(t) > 0$, $v(t) > 0$ for all $t \in [0,T)$. Then we give sufficient conditions for the nonexistence of positive solutions for this problem. Similar results for other three boundary value problems are also presented. System (S) with the multi-point boundary conditions $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, $u(T) = \sum_{i=1}^{p-2} a_i u(\xi_i) + a_0$, $v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0$, $v(T) = \sum_{i=1}^{q-2} b_i v(\eta_i) + b_0$, $(a_0, b_0 > 0)$ has been investigated in [1].

Boundary value problems with positive solutions describe many phenomena in the applied sciences such as the nonlinear diffusion generated by nonlinear sources, thermal ignition of gases and concentration in chemical or biological problems. Problems with integral boundary conditions arise in thermal conduction problems, semiconductor problems and hydrodynamic problems. In the last decades, many authors investigated differential equations or systems of differential equations with integral boundary conditions, for which they prove the existence, multiplicity and nonexistence of positive solutions by using various methods, such as fixed point theorems in cones, the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder type, fixed point index theory and coincidence degree theory (see, for example, [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]).

The paper is organized as follows. Section 2 contains some auxiliary results. The main theorems are presented in Section 3, and in Section 4 we give an example which supports our results.

2 Auxiliary Results

In this section we present some auxiliary results from [16] related to the following n-order differential equation

$$u^{(n)}(t) + z(t) = 0, \quad t \in (0,T), \tag{2.1}$$

with the integral boundary conditions

$$u(0) = \int_0^T u(s)\,dH_1(s),\ \ u'(0) = \cdots = u^{(n-2)}(0) = 0,\ \ u(T) = \int_0^T u(s)\,dH_2(s), \tag{2.2}$$

where $n \in \mathbb{N}$, $n \geq 2$, and H_1, $H_2 : [0,T] \to \mathbb{R}$ are functions of bounded variation. If $n = 2$, the condition (2.2) has the form $u(0) = \int_0^T u(s)\,dH_1(s)$, $u(T) = \int_0^T u(s)\,dH_2(s)$.

Lemma 2.1. *([16]) If H_1, H_2 are functions of bounded variation, $\Delta_1 = \left(1 - \int_0^T dH_2(s)\right) \times \times \int_0^T s^{n-1} dH_1(s) + \left(1 - \int_0^T dH_1(s)\right)\left(T^{n-1} - \int_0^T s^{n-1} dH_2(s)\right) \neq 0$, and $z \in C[0,T]$, then the solution $u \in C^n[0,T]$ of (2.1)-(2.2) is given by $u(t) = \int_0^T G_1(t,s)z(s)\,ds$, where the Green's function G_1 is defined by*

$$\begin{aligned} G_1(t,s) &= g_1(t,s) + \frac{1}{\Delta_1}\left[\left(T^{n-1} - t^{n-1}\right)\left(1 - \int_0^T dH_2(\tau)\right)\right. \\ &+ \left.\int_0^T \left(T^{n-1} - \tau^{n-1}\right) dH_2(\tau)\right]\int_0^T g_1(\tau,s)dH_1(\tau) \\ &+ \frac{1}{\Delta_1}\left[t^{n-1}\left(1 - \int_0^T dH_1(\tau)\right) + \int_0^T \tau^{n-1} dH_1(\tau)\right]\int_0^T g_1(\tau,s)dH_2(\tau), \end{aligned} \tag{2.3}$$

for all $(t,s) \in [0,T] \times [0,T]$, and

$$g_1(t,s) = \frac{1}{(n-1)!T^{n-1}}\begin{cases} t^{n-1}(T-s)^{n-1} - T^{n-1}(t-s)^{n-1}, & 0 \leq s \leq t \leq T, \\ t^{n-1}(T-s)^{n-1}, & 0 \leq t \leq s \leq T. \end{cases} \tag{2.4}$$

Lemma 2.2. *([16]) The function g_1 given by (2.4) has the properties:*

a) $g_1 : [0,T] \times [0,T] \to \mathbb{R}$ is a continuous function, $g_1(t,s) \geq 0$ for all $(t,s) \in [0,T] \times [0,T]$, $g_1(t,s) > 0$ for all $(t,s) \in (0,T) \times (0,T)$.

b) $g_1(t,s) \leq h_1(s)$ for all $(t,s) \in [0,T] \times [0,T]$, where $h_1(s) = \frac{s(T-s)^{n-1}}{(n-2)!T}$.

c) $g_1(t,s) \geq k_1(t)h_1(s)$ for all $(t,s) \in [0,T] \times [0,T]$, where

$$k_1(t) = \min\left\{\frac{(T-t)t^{n-2}}{(n-1)T^{n-1}}, \frac{t^{n-1}}{(n-1)T^{n-1}}\right\} = \begin{cases} \dfrac{t^{n-1}}{(n-1)T^{n-1}}, & 0 \leq t \leq T/2, \\ \dfrac{(T-t)t^{n-2}}{(n-1)T^{n-1}}, & T/2 \leq t \leq T. \end{cases}$$

Lemma 2.3. *([16]) Assume that H_1, $H_2 : [0,T] \to \mathbb{R}$ are nondecreasing functions, $H_1(T) - H_1(0) < 1$ and $H_2(T) - H_2(0) < 1$. Then the Green's function G_1 of problem (2.1)-(2.2), given by (2.3), satisfies the properties*

a) $G_1 : [0,T] \times [0,T] \to \mathbb{R}$ is a continuous function, $G_1(t,s) \geq 0$ for all $(t,s) \in [0,T] \times [0,T]$, and $G_1(t,s) > 0$ for all $(t,s) \in (0,T) \times (0,T)$.

b) $G_1(t,s) \leq J_1(s)$, $\forall\,(t,s) \in [0,T] \times [0,T]$, where $J_1(s) = \tau_1 h_1(s)$, $s \in [0,T]$ and

$$\begin{aligned} \tau_1 &= 1 + \frac{1}{\Delta_1}\left[T^{n-1}(1 - H_2(T) + H_2(0)) + \int_0^T (T^{n-1} - \tau^{n-1})dH_2(\tau)\right] \\ &\times (H_1(T) - H_1(0)) + \frac{1}{\Delta_1}\left[T^{n-1}(1 - H_1(T) + H_1(0)) + \int_0^T \tau^{n-1} dH_1(\tau)\right] \\ &\times (H_2(T) - H_2(0)). \end{aligned}$$

c) $G_1(t,s) \geq \gamma_1(t) J_1(s),\ \ \forall\,(t,s) \in [0,T] \times [0,T]$, *where*

$$\gamma_1(t) = \frac{1}{\tau_1}\left\{ k_1(t) + \frac{1}{\Delta_1}\left[(T^{n-1} - t^{n-1})(1 - H_2(T) + H_2(0)) \right.\right.$$
$$\left. + \int_0^T (T^{n-1} - \tau^{n-1}) dH_2(\tau) \right] \int_0^T k_1(\tau) dH_1(\tau)$$
$$\left. + \frac{1}{\Delta_1}\left[t^{n-1}(1 - H_1(T) + H_1(0)) + \int_0^T \tau^{n-1} dH_1(\tau) \right] \int_0^T k_1(\tau) dH_2(\tau) \right\}.$$

Lemma 2.4. *([16]) Assume that $H_1,\ H_2 : [0,T] \to \mathbb{R}$ are nondecreasing functions, $H_1(T) - H_1(0) < 1$, $H_2(T) - H_2(0) < 1$, and $z \in C[0,T]$ with $z(t) \geq 0$ for all $t \in [0,T]$. Then the solution u of problem (2.1)-(2.2), given in Lemma 2.1, satisfies the inequalities $u(t) \geq 0$ and $u(t) \geq \gamma_1(t) \max_{t' \in [0,T]} u(t')$ for all $t \in [0,T]$.*

We can also formulate similar results as Lemmas 2.1-2.4 for the ordinary differential equation

$$v^{(m)}(t) + \widetilde{z}(t) = 0,\ \ 0 < t < T, \tag{2.5}$$

with the integral boundary conditions

$$v(0) = \int_0^T v(s)\, dK_1(s),\ \ v'(0) = \cdots = v^{(m-2)}(0) = 0,\ \ v(T) = \int_0^T v(s)\, dK_2(s), \tag{2.6}$$

where $m \in \mathbb{N}$, $m \geq 2$, $K_1,\ K_2 : [0,T] \to \mathbb{R}$ are nondecreasing functions and $\widetilde{z} \in C[0,T]$. In the case $m = 2$, the boundary conditions have the form $v(0) = \int_0^T v(s)\, dK_1(s)$, $v(T) = \int_0^T v(s)\, dK_2(s)$. We denote by Δ_2, g_2, G_2, h_2, k_2, τ_2, J_2 and γ_2 the corresponding constants and functions for problem (2.5)-(2.6) defined in a similar manner as Δ_1, g_1, G_1, h_1, k_1, τ_1, J_1 and γ_1, respectively.

In the proof of our existence result, we shall use the Schauder fixed point theorem which we present now.

Theorem 2.5. *Let X be a Banach space and $Y \subset X$ a nonempty, bounded, convex and closed subset. If the operator $A : Y \to Y$ is completely continuous, then A has at least one fixed point.*

3 Main Results

We present the assumptions that we shall use in the sequel.

- $(J1)$ $H_1,\ H_2,\ K_1,\ K_2 : [0,T] \to \mathbb{R}$ are nondecreasing functions, $H_1(T) - H_1(0) < 1$, $H_2(T) - H_2(0) < 1$, $K_1(T) - K_1(0) < 1$ and $K_2(T) - K_2(0) < 1$.
- $(J2)$ The functions $a,\ b : [0,T] \to [0,\infty)$ are continuous and there exist $t_1,\ t_2 \in (0,T)$ such that $a(t_1) > 0$, $b(t_2) > 0$.
- $(J3)$ $f,\ g : [0,\infty) \to [0,\infty)$ are continuous functions and there exists $c_0 > 0$ such that $f(u) < \frac{c_0}{L}$, $g(u) < \frac{c_0}{L}$ for all $u \in [0,c_0]$, where $L = \max\{\int_0^T a(s) J_1(s)\, ds, \int_0^T b(s) J_2(s)\, ds\}$ and $J_1,\ J_2$ are defined in Section 2.
- $(J4)$ $f,\ g : [0,\infty) \to [0,\infty)$ are continuous functions and satisfy the conditions
  $\lim\limits_{u\to\infty} \dfrac{f(u)}{u} = \infty,\ \ \lim\limits_{u\to\infty} \dfrac{g(u)}{u} = \infty.$

Our first theorem is the following existence result for problem $(S) - (BC)$.

Theorem 3.1. *Assume that assumptions $(J1) - (J3)$ hold. Then problem $(S) - (BC)$ has at least one positive solution for $a_0 > 0$ and $b_0 > 0$ sufficiently small.*

Proof. By $(J1)$-$(J2)$, we deduce that $\int_0^T a(s)J_1(s)\,ds > 0$ and $\int_0^T b(s)J_2(s)\,ds > 0$, that is, the constant L from $(J3)$ is positive. We consider the problems

$$\begin{cases} h^{(n)}(t) = 0, \;\; t \in (0,T), \\ h(0) = \int_0^T h(s)\,dH_1(s) + 1, \;\; h'(0) = \cdots = h^{(n-2)}(0) = 0, \;\; h(T) = \int_0^T h(s)\,dH_2(s), \end{cases} \tag{3.1}$$

$$\begin{cases} k^{(m)}(t) = 0, \;\; t \in (0,T), \\ k(0) = \int_0^T, k(s)\,dK_1(s) + 1, \;\; k'(0) = \cdots = k^{(m-2)}(0) = 0, \;\; k(T) = \int_0^T k(s)\,dK_2(s). \end{cases} \tag{3.2}$$

The above problems (3.1) and (3.2) have the solutions

$$\begin{aligned} h(t) &= \frac{1}{\Delta_1}\left[-t^{n-1}\left(1 - \int_0^T dH_2(s)\right) + T^{n-1} - \int_0^T s^{n-1}\,dH_2(s)\right], \;\; t \in [0,T], \\ k(t) &= \frac{1}{\Delta_2}\left[-t^{m-1}\left(1 - \int_0^T dK_2(s)\right) + T^{m-1} - \int_0^T s^{m-1}\,dK_2(s)\right], \;\; t \in [0,T], \end{aligned} \tag{3.3}$$

respectively, where Δ_1 and Δ_2 are defined in Section 2. By assumption $(J1)$ we obtain $h(t) > 0$ and $k(t) > 0$ for all $t \in [0,T)$.

We define the functions $x(t)$ and $y(t)$, $t \in [0,T]$ by $x(t) = u(t) - a_0 h(t)$ and $y(t) = v(t) - b_0 k(t)$ for all $t \in [0,T]$, where (u,v) is a solution of $(S)-(BC)$. Then $(S)-(BC)$ can be equivalently written as

$$\begin{cases} x^{(n)}(t) + a(t) f(y(t) + b_0 k(t)) = 0, \;\; t \in (0,T), \\ y^{(m)}(t) + b(t) g(x(t) + a_0 h(t)) = 0, \;\; t \in (0,T), \end{cases} \tag{3.4}$$

with the boundary conditions

$$\begin{cases} x(0) = \displaystyle\int_0^T x(s)\,dH_1(s), \;\; x'(0) = \cdots = x^{(n-2)}(0) = 0, \;\; x(T) = \displaystyle\int_0^T x(s)\,dH_2(s), \\ y(0) = \displaystyle\int_0^T y(s)\,dK_1(s), \;\; y'(0) = \cdots = y^{(m-2)}(0) = 0, \;\; y(T) = \displaystyle\int_0^T y(s)\,dK_2(s). \end{cases} \tag{3.5}$$

Using the Green's functions G_1 and G_2 from Section 2, we find a pair (x,y) is a solution of problem (3.4)-(3.5) if and only if (x,y) is a solution for the nonlinear integral equations

$$\begin{cases} x(t) = \displaystyle\int_0^T G_1(t,s) a(s) f\left(\int_0^T G_2(s,\tau) b(\tau) g(x(\tau) + a_0 h(\tau))\,d\tau + b_0 k(s)\right) ds, \\ y(t) = \displaystyle\int_0^T G_2(t,s) b(s) g(x(s) + a_0 h(s))\,ds, \;\; 0 \le t \le T, \end{cases} \tag{3.6}$$

where $h(t), \; k(t), \; t \in [0,T]$ are given by (3.3).

We consider the Banach space $X = C[0,T]$ with the supremum norm $\|\cdot\|$ and define the set $E = \{x \in C[0,T], \;\; 0 \le x(t) \le c_0, \;\; \forall\, t \in [0,T]\} \subset X$.

We also define the operator $\mathcal{A} : E \to X$ by

$$(\mathcal{A}x)(t) = \int_0^T G_1(t,s) a(s) f\left(\int_0^T G_2(s,\tau) b(\tau) g(x(\tau) + a_0 h(\tau)) d\tau + b_0 k(s)\right) ds,$$
$$0 \le t \le T, \;\; x \in E.$$

For sufficiently small $a_0 > 0$ and $b_0 > 0$, by $(J3)$, we deduce that $f(y(t) + b_0 k(t)) \le \frac{c_0}{L}$ and $g(x(t) + a_0 h(t)) \le \frac{c_0}{L}$ for all $t \in [0,T]$ and $x, y \in E$. Then, by using Lemma 2.4, we obtain $(\mathcal{A}x)(t) \ge 0$ for all $t \in [0,T]$ and $x \in E$.

By Lemma 2.3, for all $x \in E$, we have

$$\int_0^T G_2(s,\tau)b(\tau)g(x(\tau)+a_0h(\tau))\,d\tau \le \int_0^T J_2(\tau)b(\tau)g(x(\tau)+a_0h(\tau))\,d\tau$$
$$\le \frac{c_0}{L}\int_0^T J_2(\tau)b(\tau)\,d\tau \le c_0,\ \ \forall\, s\in[0,T],$$

and

$$(\mathcal{A}x)(t) \le \int_0^T J_1(s)a(s)f\left(\int_0^T G_2(s,\tau)b(\tau)g(x(\tau)+a_0h(\tau))d\tau + b_0k(s)\right)ds$$
$$\le \frac{c_0}{L}\int_0^T J_1(s)a(s)\,ds \le c_0,\ \ \forall\, t\in[0,T].$$

Therefore $\mathcal{A}(E)\subset E$.

Using standard arguments, we deduce that $\mathcal{A}$ is completely continuous. By Theorem 2.5, we conclude that $\mathcal{A}$ has a fixed point $x\in E$. This element together with y given by (3.6) represents a solution for (3.4)-(3.5). This shows that our problem $(S)-(BC)$ has a positive solution (u,v) with $u = x + a_0h,\ v = y + b_0k$ for sufficiently small a_0 and b_0. □

In what follows, we present sufficient conditions for the nonexistence of positive solutions of $(S)-(BC)$.

Theorem 3.2. *Assume that assumptions $(J1)$, $(J2)$ and $(J4)$ hold. Then problem $(S)-(BC)$ has no positive solution for a_0 and b_0 sufficiently large.*

Proof. We suppose that (u,v) is a positive solution of $(S)-(BC)$. Then (x,y) with $x = u - a_0h$, $y = v - b_0k$ is a solution for (3.4)-(3.5), where h and k are the solutions of problems (3.1) and (3.2), respectively, (given by (3.3)). By $(J2)$ there exists $c\in(0,T/2)$ such that $t_1,\ t_2\in(c,T-c)$, and then $\int_c^{T-c} a(s)J_1(s)\,ds > 0$, $\int_c^{T-c} b(s)J_2(s)\,ds > 0$. Now by using Lemma 2.4, we have $x(t)\ge 0$, $y(t)\ge 0$ for all $t\in[0,T]$, and $\inf_{t\in[c,T-c]} x(t) \ge \gamma_1^0\|x\|$ and $\inf_{t\in[c,T-c]} y(t)\ge \gamma_2^0\|y\|$, where $\gamma_1^0 = \inf_{t\in[c,T-c]}\gamma_1(t)$, $\gamma_2^0 = \inf_{t\in[c,T-c]}\gamma_2(t)$.

Using now (3.3), we deduce that

$$\inf_{t\in[c,T-c]} h(t) = h(T-c) = \frac{h(T-c)}{h(0)}\|h\|,\quad \inf_{t\in[c,T-c]} k(t) = k(T-c) = \frac{k(T-c)}{k(0)}\|k\|.$$

Therefore, we obtain

$$\inf_{t\in[c,T-c]}(x(t)+a_0h(t)) \ge \gamma_1^0\|x\| + a_0\frac{h(T-c)}{h(0)}\|h\| \ge r_1(\|x\|+a_0\|h\|) \ge r_1\|x+a_0h\|,$$
$$\inf_{t\in[c,T-c]}(y(t)+b_0k(t)) \ge \gamma_2^0\|y\| + b_0\frac{k(T-c)}{k(0)}\|k\| \ge r_2(\|y\|+b_0\|k\|) \ge r_2\|y+b_0k\|,$$

where $r_1 = \min\left\{\gamma_1^0, \frac{h(T-c)}{h(0)}\right\}$, $r_2 = \min\left\{\gamma_2^0, \frac{k(T-c)}{k(0)}\right\}$.
We now consider $R = \left(\min\left\{\gamma_2^0 r_1\int_c^{T-c} b(s)J_2(s)\,ds,\ \gamma_1^0 r_2\int_c^{T-c} a(s)J_1(s)\,ds\right\}\right)^{-1} > 0$.

By using $(J4)$, for R defined above, we conclude that there exists $M>0$ such that $f(u) > 2Ru,\ g(u) > 2Ru$ for all $u\ge M$. We consider $a_0>0$ and $b_0>0$ sufficiently large such that $\inf_{t\in[c,T-c]}(x(t)+a_0h(t))\ge M$ and $\inf_{t\in[c,T-c]}(y(t)+b_0k(t))\ge M$. By $(J2)$, (3.4), (3.5) and the above inequalities, we deduce that $\|x\|>0$ and $\|y\|>0$.

Now by using Lemma 2.3 and the above considerations, we have

$$
\begin{aligned}
y(c) &= \int_0^T G_2(c,s)b(s)g(x(s)+a_0h(s))\,ds \geq \int_0^T \gamma_2(c)J_2(s)b(s)g(x(s)+a_0h(s))\,ds \\
&\geq \gamma_2^0 \int_c^{T-c} J_2(s)b(s)g(x(s)+a_0h(s))\,ds \geq 2R\gamma_2^0 \int_c^{T-c} J_2(s)b(s)(x(s)+a_0h(s))\,ds \\
&\geq 2R\gamma_2^0 \int_c^{T-c} J_2(s)b(s) \inf_{\tau\in[c,T-c]}(x(\tau)+a_0h(\tau))\,ds \\
&\geq 2R\gamma_2^0 r_1 \int_c^{T-c} J_2(s)b(s)\|x+a_0h\|\,ds \geq 2\|x+a_0h\| \geq 2\|x\|.
\end{aligned}
$$

Therefore, we obtain

$$\|x\| \leq y(c)/2 \leq \|y\|/2. \tag{3.7}$$

In a similar manner, we deduce

$$
\begin{aligned}
x(c) &= \int_0^T G_1(c,s)a(s)f(y(s)+b_0k(s))\,ds \geq \int_0^T \gamma_1(c)J_1(s)a(s)f(y(s)+b_0k(s))\,ds \\
&\geq \gamma_1^0 \int_c^{T-c} J_1(s)a(s)f(y(s)+b_0k(s))\,ds \geq 2R\gamma_1^0 \int_c^{T-c} J_1(s)a(s)(y(s)+b_0k(s))\,ds \\
&\geq 2R\gamma_1^0 \int_c^{T-c} J_1(s)a(s) \inf_{\tau\in[c,T-c]}(y(\tau)+b_0k(\tau))\,ds \\
&\geq 2R\gamma_1^0 r_2 \int_c^{T-c} J_1(s)a(s)\|y+b_0k\|\,ds \geq 2\|y+b_0k\| \geq 2\|y\|.
\end{aligned}
$$

So, we obtain

$$\|y\| \leq x(c)/2 \leq \|x\|/2. \tag{3.8}$$

By (3.7) and (3.8), we conclude that $\|x\| \leq \|y\|/2 \leq \|x\|/4$, which is a contradiction, because $\|x\| > 0$. Then, for a_0 and b_0 sufficiently large, our problem $(S)-(BC)$ has no positive solution. □

Similar results as Theorems 3.1 and 3.2 can be obtained if instead of boundary conditions (BC) we have

$$
(BC_1)\quad \begin{cases} u(0) = \int_0^T u(s)\,dH_1(s), \;\; u'(0) = \cdots = u^{(n-2)}(0) = 0, \;\; u(T) = \int_0^T u(s)\,dH_2(s) + a_0, \\ v(0) = \int_0^T v(s)\,dK_1(s), \;\; v'(0) = \cdots = v^{(m-2)}(0) = 0, \;\; v(T) = \int_0^T v(s)\,dK_2(s) + b_0, \end{cases}
$$

or

$$
(BC_2)\quad \begin{cases} u(0) = \int_0^T u(s)\,dH_1(s) + a_0, \;\; u'(0) = \cdots = u^{(n-2)}(0) = 0, \;\; u(T) = \int_0^T u(s)\,dH_2(s), \\ v(0) = \int_0^T v(s)\,dK_1(s), \;\; v'(0) = \cdots = v^{(m-2)}(0) = 0, \;\; v(T) = \int_0^T v(s)\,dK_2(s) + b_0, \end{cases}
$$

or

$$
(BC_3)\quad \begin{cases} u(0) = \int_0^T u(s)\,dH_1(s), \;\; u'(0) = \cdots = u^{(n-2)}(0) = 0, \;\; u(T) = \int_0^T u(s)\,dH_2(s) + a_0, \\ v(0) = \int_0^T v(s)\,dK_1(s) + b_0, \;\; v'(0) = \cdots = v^{(m-2)}(0) = 0, \;\; v(T) = \int_0^T v(s)\,dK_2(s), \end{cases}
$$

where a_0 and b_0 are positive constants.

For problem $(S)-(BC_1)$, instead of functions h and k from the proof of Theorem 3.1, the solutions of problems

$$\begin{cases} \widetilde{h}^{(n)}(t)=0, \ \ t\in(0,T), \\ \widetilde{h}(0)=\int_0^T \widetilde{h}(s)\,dH_1(s), \ \ \widetilde{h}'(0)=\cdots=\widetilde{h}^{(n-2)}(0)=0, \ \ \widetilde{h}(T)=\int_0^T \widetilde{h}(s)\,dH_2(s)+1, \end{cases} \tag{3.9}$$

$$\begin{cases} \widetilde{k}^{(m)}(t)=0, \ \ t\in(0,T), \\ \widetilde{k}(0)=\int_0^T \widetilde{k}(s)\,dK_1(s), \ \ \widetilde{k}'(0)=\cdots=\widetilde{k}^{(m-2)}(0)=0, \ \ \widetilde{k}(T)=\int_0^T \widetilde{k}(s)\,dK_2(s)+1 \end{cases} \tag{3.10}$$

are

$$\widetilde{h}(t)=\frac{1}{\Delta_1}\left[t^{n-1}\left(1-\int_0^T dH_1(s)\right)+\int_0^T s^{n-1}\,dH_1(s)\right], \ \ t\in[0,T],$$
$$\widetilde{k}(t)=\frac{1}{\Delta_2}\left[t^{m-1}\left(1-\int_0^T dK_1(s)\right)+\int_0^T s^{m-1}\,dK_1(s)\right], \ \ t\in[0,T],$$

respectively. By assumption $(J1)$ we obtain $\widetilde{h}(t)>0$ and $\widetilde{k}(t)>0$ for all $t\in(0,T]$.

For problem $(S)-(BC_2)$, instead of functions h and k from the proof of Theorem 3.1, the solutions of problems (3.1) and (3.10) are the functions h and $\widetilde{k}$, respectively, which satisfy $h(t)>0$ for all $t\in[0,T)$ and $\widetilde{k}(t)>0$ for all $t\in(0,T]$. For problem $(S)-(BC_3)$, instead of functions h and k from the proof of Theorem 3.1, the solutions of problems (3.9) and (3.2) are the functions $\widetilde{h}$ and k, respectively, which satisfy $\widetilde{h}(t)>0$ for all $t\in(0,T]$ and $k(t)>0$ for all $t\in[0,T)$.

Therefore we also obtain the following results.

Theorem 3.3. *Assume that assumptions $(J1)-(J3)$ hold. Then problem $(S)-(BC_1)$ has at least one positive solution ($u(t)>0$ and $v(t)>0$ for all $t\in(0,T]$) for $a_0>0$ and $b_0>0$ sufficiently small.*

Theorem 3.4. *Assume that assumptions $(J1)$, $(J2)$ and $(J4)$ hold. Then problem $(S)-(BC_1)$ has no positive solution ($u(t)>0$ and $v(t)>0$ for all $t\in(0,T]$) for a_0 and b_0 sufficiently large.*

Theorem 3.5. *Assume that assumptions $(J1)-(J3)$ hold. Then problem $(S)-(BC_2)$ has at least one positive solution ($u(t)>0$ for all $t\in[0,T)$, and $v(t)>0$ for all $t\in(0,T]$) for $a_0>0$ and $b_0>0$ sufficiently small.*

Theorem 3.6. *Assume that assumptions $(J1)$, $(J2)$ and $(J4)$ hold. Then problem $(S)-(BC_2)$ has no positive solution ($u(t)>0$ for all $t\in[0,T)$, and $v(t)>0$ for all $t\in(0,T]$) for a_0 and b_0 sufficiently large.*

Theorem 3.7. *Assume that assumptions $(J1)-(J3)$ hold. Then problem $(S)-(BC_3)$ has at least one positive solution ($u(t)>0$ for all $t\in(0,T]$, and $v(t)>0$ for all $t\in[0,T)$) for $a_0>0$ and $b_0>0$ sufficiently small.*

Theorem 3.8. *Assume that assumptions $(J1)$, $(J2)$ and $(J4)$ hold. Then problem $(S)-(BC_3)$ has no positive solution ($u(t)>0$ for all $t\in(0,T]$, and $v(t)>0$ for all $t\in[0,T)$) for a_0 and b_0 sufficiently large.*

4 An Example

We consider $T = 1$, $n = 3$, $m = 4$, $a(t) = at^2$, $b(t) = bt^3$, for all $t \in [0, 1]$ with a, $b > 0$, $H_1(t) = \frac{t^4}{3}$, $K_2(t) = t^3/2$, and

$$H_2(t) = \begin{cases} 0, & t \in [0, 1/3), \\ 1/3, & t \in [1/3, 2/3), \\ 5/6, & t \in [2/3, 1], \end{cases} \quad K_1(t) = \begin{cases} 0, & t \in [0, 1/2), \\ 1/2, & t \in [1/2, 1]. \end{cases}$$

Then, we have $\int_0^1 u(s)dH_1(s) = \frac{4}{3}\int_0^1 s^3u(s)\,ds$, $\int_0^1 u(s)dH_2(s) = \frac{1}{3}u\left(\frac{1}{3}\right) + \frac{1}{2}u\left(\frac{2}{3}\right)$, $\int_0^1 v(s)dK_1(s) = \frac{1}{2}v\left(\frac{1}{2}\right)$, $\int_0^1 v(s)dK_2(s) = \frac{3}{2}\int_0^1 s^2v(s)\,ds$. We also consider the functions f, $g : [0,\infty) \to [0,\infty)$, $f(x) = \frac{\widetilde{a}x^\alpha}{x^\beta + \widetilde{c}}$, $g(x) = \frac{\widetilde{b}x^\gamma}{x^\delta + \widetilde{d}}$ for all $x \in [0,\infty)$, with $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{d} > 0$, α, β, γ, $\delta > 0$, $\alpha > \beta + 1$, and $\gamma > \delta + 1$. We have $\lim_{x\to\infty} f(x)/x = \lim_{x\to\infty} g(x)/x = \infty$.

Therefore, we consider the nonlinear higher-order differential system

$$(S_0) \qquad \begin{cases} u^{(3)}(t) + at^2\dfrac{\widetilde{a}v^\alpha(t)}{v^\beta(t) + \widetilde{c}} = 0, \ \ t \in (0, 1), \\ v^{(4)}(t) + bt^3\dfrac{\widetilde{b}u^\gamma(t)}{u^\delta(t) + \widetilde{d}} = 0, \ \ t \in (0, 1), \end{cases}$$

with the boundary conditions

$$(BC_0) \qquad \begin{cases} u(0) = \frac{4}{3}\int_0^1 s^3u(s)\,ds + a_0, \ \ u'(0) = 0, \ \ u(1) = \frac{1}{3}u\left(\frac{1}{3}\right) + \frac{1}{2}u\left(\frac{2}{3}\right), \\ v(0) = \frac{1}{2}v\left(\frac{1}{2}\right) + b_0, \ \ v'(0) = v''(0) = 0, \ \ v(1) = \frac{3}{2}\int_0^1 s^2v(s)\,ds. \end{cases}$$

Then, we obtain $H_1(1) - H_1(0) = \frac{1}{3} < 1$, $H_2(1) - H_2(0) = \frac{5}{6} < 1$, $K_1(1) - K_1(0) = \frac{1}{2} < 1$ and $K_2(1) - K_2(0) = \frac{1}{2} < 1$. We deduce that assumptions $(J1)$, $(J2)$ and $(J4)$ are satisfied. We also obtain $\Delta_1 = \frac{43}{81}$, $\Delta_2 = \frac{13}{32}$, $\tau_1 = \frac{123}{43}$, $\tau_2 = \frac{34}{13}$, $h_1(s) = s(1-s)^2$, $h_2(s) = \frac{1}{2}s(1-s)^3$, $J_1(s) = \frac{123}{43}s(1-s)^2$, and $J_2(s) = \frac{17}{13}s(1-s)^3$, $\ s \in [0, 1]$.

By using the above functions J_1 and J_2, we deduce $\widetilde{A} = \int_0^1 s^2J_1(s)\,ds \approx 0.04767442$, $\widetilde{B} = \int_0^1 s^3J_2(s)\,ds \approx 0.00467033$, and then $L = \max\{a\widetilde{A}, b\widetilde{B}\}$. We choose $c_0 = 1$ and if we select $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{d}$ satisfying the conditions $\widetilde{a} < \frac{1+\widetilde{c}}{L} = (1+\widetilde{c})\min\left\{\frac{1}{a\widetilde{A}}, \frac{1}{b\widetilde{B}}\right\}$, $\widetilde{b} < \frac{1+\widetilde{d}}{L} = (1+\widetilde{d})\min\left\{\frac{1}{a\widetilde{A}}, \frac{1}{b\widetilde{B}}\right\}$, then we conclude that $f(x) \le \frac{\widetilde{a}}{1+\widetilde{c}} < \frac{1}{L}$, $g(x) \le \frac{\widetilde{b}}{1+\widetilde{d}} < \frac{1}{L}$ for all $x \in [0, 1]$. For example, if $a = 2$, $b = 3$, $\widetilde{c} = \widetilde{d} = 1$, then for $\widetilde{a} \le 20.97$ and $\widetilde{b} \le 20.97$ the above conditions for f and g are satisfied. So, assumption $(J3)$ is also satisfied. By Theorems 3.1 and 3.2 we deduce that problem $(S_0) - (BC_0)$ has at least one positive solution (here $u(t) > 0$ and $v(t) > 0$ for all $t \in [0, 1]$) for sufficiently small $a_0 > 0$ and $b_0 > 0$, and no positive solution for sufficiently large a_0 and b_0.

5 Conclusion

In this paper, we studied the system of nonlinear higher-order ordinary differential equations (S) with the Riemann-Stieltjes integral boundary conditions (BC) which contain some positive constants. By using the Schauder fixed point theorem and some properties of the associated Green's functions, we show that this problem has at least one positive solution for sufficiently small constants. Then, we give sufficient conditions for the nonexistence of positive solutions. Similar results for other three boundary value problems are also presented.

Competing Interests

Authors have declared that no competing interests exist.

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