British Journal of Mathematics & Computer Science
11(3): 1-8, 2015, Article no.BJMCS.20271
ISSN: 2231-0851

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Oscillation Criteria for Higher Order Nonlinear Functional Difference Equations

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Article Information

DOI: 10.9734/BJMCS/2015/20271
Editor(s):
(1) Sergio Serrano, Department of Applied Mathematics, University of Zaragoza, Spain.
Reviewers:
(1) Anonymous, Universidad Autnoma Metropolitana, Mexico.
(2) Anonymous, Neijiang Normal University, China.
(3) Anonymous, Chang-Jung Christian University, Taiwan.
Complete Peer review History: http://sciencedomain.org/review-history/11225

Original Research Article

Received: 20 July 2015
Accepted: 16 August 2015
Published: 01 September 2015

Abstract

In this paper some criteria for the oscillation of high order functional difference equation of the form

$$\Delta^2 \left(r(n) \left[\Delta^{(m-2)} y(n) \right]^{\alpha} \right) + q(n) f[y(g(n))] = 0,$$

where $\sum_{s=n_0}^{\infty} r^{-\frac{1}{\alpha}}(s) < \infty$ and $m > 1$
is discussed. Examples are given to illustrate the results.

Keywords: Functional difference equation; Nonlinear; non-oscillation; oscillation.

2010 Mathematics Subject Classification: 39A10.

1 Introduction

The notion of nonlinear difference equation was studied intensively by R. P. Agarwal [1] and oscillatory properties were discussed by R. P. Agarwal et al.[2, 3, 4]. Difference equations find

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a lot of applications in the natural sciences, technology and population dynamics [5, 6, 7].

Recently there has been a lot of interest in the study of oscillatory behavior of solutions of nonlinear difference equations. We can see this in [8-13]. Researchers carried out their researches on the oscillatory and asymptotic behavior of solutions of linear higher order difference equations. During the last two decades, many authors all over the world have taken keen interest in studying

oscillatory behavior of solutions of functional difference equations, due to its important applications in the field of science and computers. Such equations have extensive application in economics, physics, chemical technology, medicine, dynamic systems, optimal control and many other fields. So, we consider the high order nonlinear functional difference equation of the form

$$\Delta^2 \left(r(n) \left[\Delta^{(m-2)} y(n) \right]^{\alpha} \right) + q(n) f\left[y(g(n)) \right] = 0, \tag{1.1}$$

where

$$\sum_{s=n_0}^{\infty} r^{-\frac{1}{\alpha}}(s) < \infty. \tag{1.2}$$

subject to the hypotheses:

(i) α is the ratio of any two positive odd integers.

(ii) $\{r(n)\}, \{q(n)\}$ are real-valued positive sequences.

(iii) $\{g(n)\}$ is a realvalued increasing sequence with $g(n) < n$, for $n \geq n_0$ and $\lim_{n\to\infty} g(n) = \infty$.

(iv) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $xf(x) > 0$ and $f'(x) \geq 0$, for $x \neq 0$ and

$$-f(-xy) \geq f(xy) \geq f(x) f(y), \text{ for } xy > 0. \tag{1.3}$$

Here Δ is the forward difference operator defined by

$$\Delta y(n) = y(n+1) - y(n).$$

By a solution of equation (1.1), we mean a real sequence $\{y(n)\}$ which is defined for $n \geq \min_{i\geq 0} \{g(i)\}$ and satisfies equation (1.1) for all large n. A solution $\{y(n)\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called non-oscillatory. A difference equation is said to be oscillatory if all of its solutions are oscillatory. Otherwise, it is non-oscillatory.

Our aim in this paper is to obtain some oscillation criteria for the solution of equation (1.1). For recent contributions to this study, we refer the reader to the papers [8-13] and the references cited therein.

2 Main Results

In this section, we are interested to present some sufficient conditions for the oscillation of all solutions of the equation (1.1).

Theorem 2.1. *Assume that conditions (i)-(iv), (1.2) and (1.3) hold. If the first order delay difference equation*

$$\Delta u(n) + cq(n) f\left[(g(n) - n_2)^{m-3} \right] f\left[\sum_{s=n_2}^{g(n)-1} \left(\frac{s}{r(s)} \right)^{\frac{1}{\alpha}} \right] f\left[\left(u^{\frac{1}{\alpha}}(g(n)) \right) \right] = 0, \tag{2.1}$$

for any constant $c, 0 < c < 1$ *and for all* $N \geq n_0$ *is oscillatory, then the equation (1.1) is oscillatory.*

Proof. Let $\{y(n)\}$ be a non-oscillatory solution of equation (1.1). Without loss of generality, assume that $y(n) > 0$ and $y(g(n)) > 0$, for $n \geq n_0 \geq 0$.
Then from (1.1), we can see that

$$\Delta^2 \left(r(n) \left(\Delta^{(m-2)} y(n) \right)^{\alpha} \right) \leq 0, \text{ for } n \geq n_0. \tag{2.2}$$

There exists a $n_1 \geq n_0$ such that $\left\{ \Delta^{(m-2)} y(n) \right\}$ and $\left\{ \Delta \left(r(n) \left(\Delta^{(m-2)} y(n) \right)^{\alpha} \right) \right\}$ are eventually monotone and one-signed, for $n \geq n_1$.

We consider the following four cases for $n \geq n_1$ and $m > 1$.

(i) $\Delta \left(r(n) \left(\Delta^{(m-2)} y(n) \right)^{\alpha} \right) > 0$ and $\Delta^{(m-2)} y(n) < 0$, for $n \geq n_1$.

(ii) $\Delta \left(r(n) \left(\Delta^{(m-2)} y(n) \right)^{\alpha} \right) > 0$ and $\Delta^{(m-2)} y(n) > 0$, for $n \geq n_1$.

(iii) $\Delta \left(r(n) \left(\Delta^{(m-2)} y(n) \right)^{\alpha} \right) < 0$ and $\Delta^{(m-2)} y(n) > 0$, for $n \geq n_1$.

(iv) $\Delta \left(r(n) \left(\Delta^{(m-2)} y(n) \right)^{\alpha} \right) < 0$ and $\Delta^{(m-2)} y(n) < 0$, for $n \geq n_1$.

Case (i): Since $r(n) \left(\Delta^{(m-2)} y(n) \right)^{\alpha}$ is increasing, for $n \geq s \geq n_1$, we obtain

$$r(n) \left(\Delta^{(m-2)} y(n) \right)^{\alpha} \geq r(s) \left(\Delta^{(m-2)} y(s) \right)^{\alpha}.$$

That is,

$$r(s) \left(\Delta^{(m-2)} y(s) \right)^{\alpha} \leq r(n) \left(\Delta^{(m-2)} y(n) \right)^{\alpha},$$

which implies,

$$\Delta^{(m-2)} y(s) \leq \left[r(n) \left(\Delta^{(m-2)} y(n) \right)^{\alpha} \right]^{\frac{1}{\alpha}} \left(\frac{1}{r(s)} \right)^{\frac{1}{\alpha}}.$$

Summing the above inequality from n_1 to $n-1$, we get

$$\Delta^{(m-3)} y(n) - \Delta^{(m-3)} y(n_1) \leq \left[r(n) \left(\Delta^{(m-2)} y(n) \right)^{\alpha} \right]^{\frac{1}{\alpha}} \sum_{s=n_1}^{n-1} \left(\frac{1}{r(s)} \right)^{\frac{1}{\alpha}}.$$

That is,

$$\Delta^{(m-3)} y(n) \leq \left[r(n) \left(\Delta^{(m-2)} y(n) \right)^{\alpha} \right]^{\frac{1}{\alpha}} \sum_{s=n_1}^{n-1} \left(\frac{1}{r(s)} \right)^{\frac{1}{\alpha}} + \Delta^{(m-3)} y(n_1).$$

Since $\Delta^{(m-2)} y(n) < 0$, we have

$$\begin{aligned} \Delta^{(m-3)} y(n) &\leq \left[r(n) \left(\Delta^{(m-2)} y(n) \right)^{\alpha} \right]^{\frac{1}{\alpha}} \sum_{s=n_1}^{n-1} \left(\frac{1}{r(s)} \right)^{\frac{1}{\alpha}} \\ &= \left[r(n) \left(\Delta^{(m-2)} y(n) \right)^{\alpha} \right]^{\frac{1}{\alpha}} (n - n_1) \left(\frac{1}{r(s)} \right)^{\frac{1}{\alpha}}. \end{aligned}$$

That is,

$$\Delta^{(m-3)}y(n) \leq \left[r(n)\left(\Delta^{(m-2)}y(n)\right)^{\alpha}\right]^{\frac{1}{\alpha}}(n-n_1)\left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}}.$$

Similarly, we can obtain

$$\Delta^{(m-4)}y(n) \leq \left[r(n)\left(\Delta^{(m-2)}y(n)\right)^{\alpha}\right]^{\frac{1}{\alpha}}(n-n_1)^2\left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}}.$$

Proceeding like this, we get

$$\Delta y(n) \leq \left[r(n)\left(\Delta^{(m-2)}y(n)\right)^{\alpha}\right]^{\frac{1}{\alpha}}(n-n_1)^{m-3}\left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}}$$

and

$$y(n) \leq \left[r(n)\left(\Delta^{(m-2)}y(n)\right)^{\alpha}\right]^{\frac{1}{\alpha}}(n-n_1)^{m-3}\sum_{s=n_1}^{n-1}\left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}}. \tag{2.3}$$

From (1.2) and since $\Delta^{(m-2)}y(n) < 0$, we can see that $y(n) \to -\infty$ as $n \to \infty$, which contradicts our assumption that $y(n) > 0$.

Case (ii):
Let

$$z(n) = r(n)\left(\Delta^{(m-2)}y(n)\right)^{\alpha}.$$

Then $z(n) > 0$. Since $\Delta z(n)$ is decreasing (from (2.2)), for $n \geq n_1$, we have

$$z(n) - z(n_1) = \sum_{s=n_1}^{n-1}\Delta z(s) \geq (n-n_1)\Delta z(n).$$

Then there exists a $n_2 \geq n_1$ and a constant $b, 0 < b < 1$ such that

$$z(n) \geq bn\Delta z(n), \text{ for } n \geq n_2,$$

which implies

$$r(n)\left(\Delta^{(m-2)}y(n)\right)^{\alpha} \geq bn\Delta z(n).$$

That is,

$$\Delta^{(m-2)}y(n) \geq b^{\frac{1}{\alpha}}\left(\frac{n}{r(n)}\right)^{\frac{1}{\alpha}}(\Delta z(n))^{\frac{1}{\alpha}}, \text{ for } n \geq n_2.$$

Summing the above inequality from n_2 to $n-1$, we get

$$\Delta^{(m-3)}y(n) \geq b^{\frac{1}{\alpha}}(n-n_2)\left(\frac{s}{r(s)}\right)^{\frac{1}{\alpha}}(\Delta z(n))^{\frac{1}{\alpha}}.$$

Similarly, we can obtain

$$\Delta y(n) \geq b^{\frac{1}{\alpha}}(n-n_2)^{m-3}\left(\frac{s}{r(s)}\right)^{\frac{1}{\alpha}}(\Delta z(n))^{\frac{1}{\alpha}}.$$

Also,

$$y(n) \geq b^{\frac{1}{\alpha}} (n - n_2)^{m-3} \sum_{s=n_2}^{n-1} \left(\frac{s}{r(s)} \right)^{\frac{1}{\alpha}} (\Delta z(n))^{\frac{1}{\alpha}}.$$

There exists a $n_3 \geq n_2$ such that

$$y(g(n)) \geq b^{\frac{1}{\alpha}} (g(n) - n_2)^{m-3} \sum_{s=n_2}^{g(n)-1} \left(\frac{s}{r(s)} \right)^{\frac{1}{\alpha}} (\Delta z(g(n)))^{\frac{1}{\alpha}}, \text{ for } n \geq n_3. \tag{2.4}$$

Using (2.4) and (1.3) in equation (1.1), we obtain

$$\begin{aligned} -\Delta^2 z(n) &= q(n) f(y(g(n))) \\ &\geq f\left[b^{\frac{1}{\alpha}}\right] q(n) f\left[(g(n) - n_2)^{m-3}\right] \\ &\quad f\left[\sum_{s=n_2}^{g(n)-1} \left(\frac{s}{r(s)} \right)^{\frac{1}{\alpha}}\right] f\left[(\Delta z(g(n)))^{\frac{1}{\alpha}}\right], \text{ for } n \geq n_3. \end{aligned}$$

Setting $u(n) = \Delta z(n) > 0$, for $n \geq n_3$ in the above inequality, we get

$$\begin{aligned} -\Delta u(n) \geq\ & f\left[b^{\frac{1}{\alpha}}\right] q(n) f\left[(g(n) - n_2)^{m-3}\right] \\ & f\left[\sum_{s=n_2}^{g(n)-1} \left(\frac{s}{r(s)} \right)^{\frac{1}{\alpha}}\right] f\left[\left(u^{\frac{1}{\alpha}}(g(n))\right)\right]. \end{aligned}$$

That is,

$$\Delta u(n) + cq(n) f\left[(g(n) - n_2)^{m-3}\right] f\left[\sum_{s=n_2}^{g(n)-1} \left(\frac{s}{r(s)} \right)^{\frac{1}{\alpha}}\right] f\left[\left(u^{\frac{1}{\alpha}}(g(n))\right)\right] \leq 0, \tag{2.5}$$

where $c = f\left(b^{\frac{1}{\alpha}}\right)$, for $n \geq n_3$. Now the inequality (2.5) has eventually positive solution $u(n)$.

By a well-known result (see [10], [13]), the difference equation (2.1) also has an eventually positive solution which contradicts our assumption that (2.1) is oscillatory.

Case (iii):
This case cannot hold.
Let

$$z(n) = r(n) \left(\Delta^{(m-2)} y(n)\right)^{\alpha}.$$

Then $\Delta z(n) < 0$ and from (2.2), $\Delta^2 z(n) < 0$.
Therefore

$$\lim_{n \to \infty} z(n) = -\infty,$$

which is a contradiction to the fact that $\Delta^{(m-2)} y(n) > 0$.

Case (iv):
Since $r(n) \left(\Delta^{(m-2)} y(n)\right)^{\alpha}$ is decreasing, for $s \geq n \geq n_1$, we get

$$r(s) \left(\Delta^{(m-2)} y(s)\right)^{\alpha} \leq r(n) \left(\Delta^{(m-2)} y(n)\right)^{\alpha},$$

which implies,

$$\Delta^{(m-2)}y\left(s\right) \le \left[r\left(n\right)\left(\Delta^{(m-2)}y\left(n\right)\right)^{\alpha}\right]^{\frac{1}{\alpha}}\left(\frac{1}{r\left(s\right)}\right)^{\frac{1}{\alpha}}.$$

Summing the above inequality from n_1 to $n-1$, we get

$$\Delta^{(m-3)}y\left(n\right) - \Delta^{(m-3)}y\left(n_1\right) \le \left[r\left(n\right)\left(\Delta^{(m-2)}y\left(n\right)\right)^{\alpha}\right]^{\frac{1}{\alpha}}\sum_{s=n_1}^{n-1}\left(\frac{1}{r\left(s\right)}\right)^{\frac{1}{\alpha}}.$$

That is,

$$\Delta^{(m-3)}y\left(n\right) \le \left[r\left(n\right)\left(\Delta^{(m-2)}y\left(n\right)\right)^{\alpha}\right]^{\frac{1}{\alpha}}\sum_{s=n_1}^{n-1}\left(\frac{1}{r\left(s\right)}\right)^{\frac{1}{\alpha}} + \Delta^{(m-3)}y\left(n_1\right).$$

Since $\Delta^{(m-2)}y\left(n\right) < 0$, we have

$$\begin{aligned}\Delta^{(m-3)}y\left(n\right) &\le \left[r\left(n\right)\left(\Delta^{(m-2)}y\left(n\right)\right)^{\alpha}\right]^{\frac{1}{\alpha}}\sum_{s=n_1}^{n-1}\left(\frac{1}{r\left(s\right)}\right)^{\frac{1}{\alpha}}\\ &= \left[r\left(n\right)\left(\Delta^{(m-2)}y\left(n\right)\right)^{\alpha}\right]^{\frac{1}{\alpha}}\left(n-n_1\right)\left(\frac{1}{r\left(s\right)}\right)^{\frac{1}{\alpha}}.\end{aligned}$$

That is,

$$\Delta^{(m-3)}y\left(n\right) \le \left[r\left(n\right)\left(\Delta^{(m-2)}y\left(n\right)\right)^{\alpha}\right]^{\frac{1}{\alpha}}\left(n-n_1\right)\left(\frac{1}{r\left(s\right)}\right)^{\frac{1}{\alpha}}.$$

Similarly, we can obtain

$$\Delta^{(m-4)}y\left(n\right) \le \left[r\left(n\right)\left(\Delta^{(m-2)}y\left(n\right)\right)^{\alpha}\right]^{\frac{1}{\alpha}}\left(n-n_1\right)^{2}\left(\frac{1}{r\left(s\right)}\right)^{\frac{1}{\alpha}}.$$

Proceeding like this, we get

$$\Delta y\left(n\right) \le \left[r\left(n\right)\left(\Delta^{(m-2)}y\left(n\right)\right)^{\alpha}\right]^{\frac{1}{\alpha}}\left(n-n_1\right)^{m-3}\left(\frac{1}{r\left(s\right)}\right)^{\frac{1}{\alpha}}$$

and

$$y\left(n\right) \le \left[r\left(n\right)\left(\Delta^{(m-2)}y\left(n\right)\right)^{\alpha}\right]^{\frac{1}{\alpha}}\left(n-n_1\right)^{m-3}\sum_{s=n_1}^{n-1}\left(\frac{1}{r\left(s\right)}\right)^{\frac{1}{\alpha}}. \tag{2.6}$$

From (1.2) and since $\Delta^{(m-2)}y\left(n\right) < 0$, we can see that $y\left(n\right) \to -\infty$ as $n \to \infty$, which contradicts our assumption that $y\left(n\right) > 0$. This completes the proof. □

The following examples are illustrative.

3 Example

Example 3.1. *Consider the fourth order equation*

$$\Delta^{2}\left(n^{6}\left(\Delta^{2}y\left(n\right)\right)^{3}\right) + 64\left[\left(n+2\right)^{6} - n^{6}\right]y^{3}\left(n-2\right) = 0. \tag{3.1}$$

Here $r(n) = n^6, \alpha = 3$ *and* $m = 4$.
Also,

$$\sum_{s=n_0}^{n-1} r^{-\frac{1}{\alpha}}(s) = \sum_{s=n_0}^{n-1} \left(n^6\right)^{-\frac{1}{3}} = \sum_{s=n_0}^{n-1} n^{-2} = \sum_{s=n_0}^{n-1} \frac{1}{n^2} < \infty$$

and $g(n) = n - 2 < n$.

We can easily see that all conditions of Theorem 2.1 are satisfied and hence all the solutions of equation (3.1) are oscillatory.

One of such solution is $y(n) = (-1)^n$.

Example 3.2. *Consider the third order equation*

$$\Delta^2 \left(n^9 (\Delta y(n))^3\right) + 216 \left[729 (n+2)^9 + 54 (n+1)^9 + n^9\right] y^3 (n-2) = 0. \tag{3.2}$$

Here $r(n) = n^9, \alpha = 3$ *and* $m = 3$.
Also,

$$\sum_{s=n_0}^{n-1} r^{-\frac{1}{\alpha}}(s) = \sum_{s=n_0}^{n-1} \left(n^9\right)^{-\frac{1}{3}} = \sum_{s=n_0}^{n-1} n^{-3} = \sum_{s=n_0}^{n-1} \frac{1}{n^3} < \infty$$

and $g(n) = n - 2 < n$.

We can easily see that all conditions of Theorem 2.1 are satisfied and hence all the solutions of equation (3.2) are oscillatory.

One of the solutions is $y(n) = \frac{(-3)^n}{2}$.

4 Conclusion

In this paper, we have proposed the comparison method for identifying oscillatory solutions of higher order functional difference equations. This method compares first order equation which is very simple. Moreover, the above examples reveal the efficiency of our method.

Acknowledgment

The authors are grateful to the reviewers and the technical program committees for the comments and suggestions.

Competing Interests

The authors declare that no competing interests exist.

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