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Group Decision Making Based on Induced Generalized Uncertain Linguistic Aggregation Operators

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Author's contribution

This whole work was carried out by the author ZZ.

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ABSTRACT

Aims: The aim of this paper is to investigate group decision making based on induced generalized uncertain linguistic aggregation operators

Study Design: In this paper, we propose some new operational laws of linguistic variables and uncertain linguistic variables on the basis of the extended triangular conorm and triangular norm, study their properties and relationships.

Place and Duration of Study: The existing operational laws of linguistic variables and uncertain linguistic variables may have some drawbacks**.**

Methodology: Based on new operational laws of linguistic variables and uncertain linguistic variables, we develop two new uncertain linguistic aggregation operators including the induced generalized uncertain linguistic ordered weighted averaging (IGULOWA) operator and induced generalized uncertain linguistic ordered weighted geometric (IGULOWG) operator.

Results: Some desirable properties and special cases of the IGULOWA and IGULOWG operators are studied, and then, the IGULOWA and IGULOWG operators are utilized to develop an approach for multiple attribute group decision making with uncertain linguistic information.

Conclusion: A practical application of the developed approach to an investment problem is given.

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Keywords: Multiple attribute group decision making; linguistic variable; uncertain linguistic variable; Induced Generalized Uncertain Linguistic Ordered Weighted Averaging (IGULOWA) operator; Induced Generalized Uncertain Linguistic Ordered Weighted Geometric (IGULOWG) operator.

1. INTRODUCTION

The aim of multiple attribute group decision making (MAGDM) is to find the most desirable alternative(s) from a given alternative set according to the preferences provided by a group of experts. In some MAGDM problems, the decision information about alternatives is usually uncertain or fuzzy due to the increasing complexity of the socio-economic environment and the vagueness of inherent subjective nature of human thinking [1-3]; thus, it may be appropriate and sufficient to assess the information in a qualitative form rather than a quantitative form. For example, when evaluating a house's cost, linguistic variables such as ''high'', ''medium'', and ''low'' are usually used, and when evaluating a house's design, linguistic variables like ''good'', ''medium'', and ''bad'' can be frequently used. In addition, in many real-word problems, the input linguistic arguments may not match any of the original linguistic labels, or may be located between two of them [4]. For example, when evaluating the ''comfort'' of a car, an expert may provide his/her opinion with uncertain linguistic variable like ''between 'fair' and 'good''' [4]. To aggregate linguistic information and uncertain linguistic information, Xu [5-8] defined some operational laws of linguistic variables and uncertain linguistic variables, and based on which, a variety of linguistic aggregation operators and uncertain linguistic aggregation operators have been developed in the past few decades, such as the linguistic weighted averaging (LWA) operator [9], the extended geometric mean (EGM) operator [10], extended ordered weighted geometric (EOWG) operator [10], extended arithmetical averaging (EAA) operator [10], extended ordered weighted averaging (EOWA) operator [10], linguistic weighted arithmetic averaging (LWAA) operator [11,12], linguistic weighted geometric averaging (LWGA) operator [5], linguistic ordered weighted geometric averaging (LOWGA) operator [5], linguistic hybrid geometric averaging (LHGA) operator [5], linguistic generalized power average (LGPA) operator [13], weighted linguistic generalized power average (WLGPA) operator [13], linguistic generalized power ordered weighted average (LGPOWA) operator [13], linguistic power ordered weighted average (LPOWA) operator [4], linguistic power ordered weighted geometric average (LPOWGA) operator [4], linguistic power ordered weighted harmonic average (LPOWHA) operator [4], linguistic power ordered weighted quadratic average (LPOWQA) operator [4], linguistic power average (LPA) operator [4], linguistic weighted PA operator [4], uncertain linguistic ordered weighted (ULOWA) operator [6], uncertain linguistic weighted averaging (ULWA) operator [6], [14], [15], uncertain linguistic hybrid aggregation (ULHA) operator [6], induced uncertain linguistic OWA (IULOWA) operator [16], uncertain linguistic geometric mean (ULGM) operator [17], uncertain linguistic weighted geometric mean (ULWGM) operator [17], uncertain linguistic ordered weighted geometric (ULOWG) operator [17], induced uncertain linguistic ordered weighted geometric (IULOWG) operator [17], uncertain linguistic PA operator [4], uncertain linguistic weighted PA operator [4], and uncertain linguistic power ordered weighted average (ULPOWA) operator [4], etc.

It is noted that the above linguistic aggregation operators and uncertain linguistic aggregation operators are constructed based on the operational laws of linguistic variables and uncertain linguistic variables proposed by Xu [2,5,7,8]. However, these operational laws may have some problems (See Remark 3.1, Example 3.1, and Example 3.3). Accordingly, the above linguistic aggregation operators and uncertain linguistic aggregation operators may have some problems (see Subsection 6.2). To overcome these issues, the extended triangular conorm (t-conorm) and triangular norm (t-norm) in $[0,t]$ are defined to deal with the linguistic information and uncertain linguistic information in Section 2. The extended t-conorm and t-norm are generalizations of many other extended t-conorms and t-norms, such as the extended Algebraic, extended Einstein, extended Hamacher and extended Frank t-conorms and t-norms. The extended t-conorm and t-norm are generated by an additive function $g(x)$ and its dual function $f(x)=g(t-x)$. When the additive generator $g(x)$ is assigned different forms, we can obtain some specific extended t-conorms and t-norms. Thus, the extended t-conorm and t-norm are more general and more flexible. Based on the extended t-conorm and t-norm, Section 3 proposes some new linguistic operational laws of linguistic variables and uncertain linguistic variables, studies their properties and relationships, and illustrates their advantages over the operational laws proposed by Xu [5-8]. Furthermore, in this section, an induced generalized uncertain linguistic ordered weighted averaging (IGULOWA) operator and an induced generalized uncertain linguistic ordered weighted geometric (IGULOWG) operator are developed based on the new operational laws. Some interesting properties and special cases of the developed operators are also investigated in the current section. We can find that the developed aggregation operators are all based on different extended t-conorms and t-norms and are used to deal with different relationships of the aggregated arguments, which can provide more choices for the decision makers. The prominent characteristic of the developed operators is that they include a variety of uncertain linguistic aggregation operators when the additive generator $g(x)$ is assigned different forms. In Section 4, an approach to multiple attribute group decision making with uncertain linguistic information is developed based on the IGULOWA and IGULOWG operators. In Section 5, a numerical example is given to illustrate the developed group decision making method. Section 6 performs a comparison analysis between our new operators and approach and other uncertain linguistic aggregation operators and MAGDM methods [4,6,1-17], and then highlights the advantages of the new operators and approach. Finally, Section 7 ends the paper with some concluding remarks.

2. PRELIMINARIES

2.1 The Fuzzy Linguistic Approach

The fuzzy linguistic approach is an approximate technique, which represents qualitative aspects as linguistic values by means of linguistic variables. Let $S=\{s_i \mid i=0,1,2,\cdots,t\}$ be a finite and totally ordered discrete linguistic term set with odd cardinality, where s_i represents a possible value for a linguistic variable. For example, a set of nine terms S could be given as follows [18-33]:

$$S=\left\{\begin{array}{l} s_0:\text{extremely poor},\ s_1:\text{very poor},\ s_2:\text{poor},\ s_3:\text{slightly poor},\ s_4:\text{fair},\ s_5:\text{slightly good},\ s_6:\text{good},\\ s_7:\text{very good},\ s_8:\text{extremely good} \end{array}\right\}.$$

Usually, it is required that linguistic term set S should satisfy the following characteristics:

(1) The set is ordered: $s_i \geq s_j$ if $i \geq j$;

(2) There is the negation operator: $\operatorname{neg}(s_i)=s_j=s_{t-i}$ such that $j=t-i$ ($t+1$ is the granularity of the term set);

(3) Max operator: $\max\left(s_i, s_j\right) = s_i$ if $s_i \geq s_j$;

(4) Min operator: $\min\left(s_i, s_j\right) = s_i$ if $s_i \leq s_j$.

To preserve all the given information, Xu [5], [7] extended the discrete linguistic term set S to a continuous linguistic term set $\overline{S} = \left\{ s_\alpha \middle| s_0 \leq s_\alpha \leq s_t, \alpha \in [0,t] \right\}$. If $s_\alpha \in S$, then s_α is called an original linguistic term, otherwise, s_α is called a virtual linguistic term. In general, the decision maker uses the original linguistic terms to evaluate alternatives, and the virtual linguistic terms can only appear in operation.

Definition 2.1. Considering any two linguistic terms $s_\alpha, s_\beta \in \overline{S}$, and $\lambda \in [0,1]$, Xu [5], [7] defined some operational laws as follows:

(1) $s_\alpha \oplus s_\beta = s_{\alpha+\beta}$;

(2) $\lambda s_\alpha = s_{\lambda\alpha}$;

(3) $s_\alpha \otimes s_\beta = s_{\alpha\beta}$;

(4) $\left(s_\alpha\right)^\lambda = s_\alpha^\lambda = s_{\alpha^\lambda}$.

In many real-word problems, the input linguistic arguments may not match any of the original linguistic labels, or may be located between two of them [4]. For example, when evaluating the ‘‘design’’ of a car, an expert may provide his/her opinion with ‘‘between ‘fair’ and ‘good’’’ [4]. To deal with such cases, Xu [6,8] defined the uncertain linguistic variables and introduced some of their operational laws.

Definition 2.2 [6,8]. Let $\tilde{s} = \left[s_\alpha, s_\beta\right]$, where $s_\alpha, s_\beta \in \overline{S}$, s_α and s_β are the lower and upper limits, respectively, we then call $\tilde{s}$ the uncertain linguistic variable.

Let $\tilde{S}$ be the set of all uncertain linguistic variables. Consider any three uncertain linguistic variables $\tilde{s} = \left[s_\alpha, s_\beta\right]$, $\tilde{s}_1 = \left[s_{\alpha_1}, s_{\beta_1}\right]$ and $\tilde{s}_2 = \left[s_{\alpha_2}, s_{\beta_2}\right]$, then their operational laws are defined as follows:

(1) $\tilde{s}_1 \oplus \tilde{s}_2 = \left[s_{\alpha_1}, s_{\beta_1}\right] \oplus \left[s_{\alpha_2}, s_{\beta_2}\right] = \left[s_{\alpha_1} \oplus s_{\alpha_2}, s_{\beta_1} \oplus s_{\beta_2}\right] = \left[s_{\alpha_1+\alpha_2}, s_{\beta_1+\beta_2}\right]$;

(2) $\lambda\tilde{s} = \lambda\left[s_\alpha, s_\beta\right] = \left[\lambda s_\alpha, \lambda s_\beta\right] = \left[s_{\lambda\alpha}, s_{\lambda\beta}\right]$, where $\lambda \in [0,1]$;

(3) $\tilde{s}_1 \otimes \tilde{s}_2 = \left[s_{\alpha_1}, s_{\beta_1}\right] \otimes \left[s_{\alpha_2}, s_{\beta_2}\right] = \left[s_{\alpha_1} \otimes s_{\alpha_2}, s_{\beta_1} \otimes s_{\beta_2}\right] = \left[s_{\alpha_1\alpha_2}, s_{\beta_1\beta_2}\right]$;

(4) $\tilde{s}^\lambda = \left[s_\alpha, s_\beta\right]^\lambda = \left[\left(s_\alpha\right)^\lambda, \left(s_\beta\right)^\lambda\right] = \left[s_{\alpha^\lambda}, s_{\beta^\lambda}\right]$, where $\lambda \in [0,1]$;

(5) $\operatorname{neg}\left(\tilde{s}\right) = \operatorname{neg}\left(\left[s_\alpha, s_\beta\right]\right) = \left[\operatorname{neg}\left(s_\beta\right), \operatorname{neg}\left(s_\alpha\right)\right] = \left[s_{t-\beta}, s_{t-\alpha}\right]$.

In order to compare uncertain linguistic variables, we give the following definitions.

Definition 2.3. For a uncertain linguistic variable $\tilde{s} = \left[s_\alpha, s_\beta \right]$, $s(\tilde{s}) = \frac{\alpha + \beta}{2t}$ is called the score function of $\tilde{s}$.

Definition 2.4. For a uncertain linguistic variable $\tilde{s} = \left[s_\alpha, s_\beta \right]$, $v(\tilde{s}) = \frac{\beta - \alpha}{2t}$ is referred to as the variance function of $\tilde{s}$.

The relationship between the score function and the variance function is similar to the relationship between the mean and variance in statistics.

Based on the score function and the variance function, we develop a comparison law to compare any two HFEs:

Definition 2.5. Let $\tilde{s}_1 = \left[s_{\alpha_1}, s_{\beta_1} \right]$ and $\tilde{s}_2 = \left[s_{\alpha_2}, s_{\beta_2} \right]$ be any two uncertain linguistic variables, and let $s(\tilde{s}_i)$ and $v(\tilde{s}_i)$ ($i = 1,2$) be the score functions and the variance functions of $\tilde{s}_i$ ($i = 1,2$), respectively. Then, the following conditions hold:

(1) If $s(\tilde{s}_1) > s(\tilde{s}_2)$, then $\tilde{s}_1 > \tilde{s}_2$.
(2) If $s(\tilde{s}_1) = s(\tilde{s}_2)$, then
 ① if $v(\tilde{s}_1) < v(\tilde{s}_2)$, then $\tilde{s}_1 > \tilde{s}_2$.
 ② if $v(\tilde{s}_1) = v(\tilde{s}_2)$, then $\tilde{s}_1 = \tilde{s}_2$.

2.2 The Extended Triangular Co Norm and Triangular Norm

Definition 2.6. A function $T : [0,t] \times [0,t] \to [0,t]$ is called an extended triangular norm (t-norm) if it satisfies the following four conditions:

(1) $T(t,a) = a$, for all $a \in [0,t]$.
(2) $T(a,b) = T(b,a)$, for all $a,b \in [0,t]$.
(3) $T(a,T(b,c)) = T(T(a,b),c)$, for all $a,b,c \in [0,t]$.
(4) If $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,t]$, then $T(a,b) \le T(c,d)$.

Definition 2.7. A function $S : [0,t] \times [0,t] \to [0,t]$ is called an extended triangular conorm (t-conorm) if it satisfies the following four conditions:

(1) $S(0,a) = a$, for all $a \in [0,t]$.
(2) $S(a,b) = S(b,a)$, for all $a,b \in [0,t]$.
(3) $S(a,S(b,c)) = S(S(a,b),c)$, for all $a,b,c \in [0,t]$.
(4) If $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,t]$, then $S(a,b) \le S(c,d)$.

The following theorem show that an extended t-norm $T(a,b)$ is expressed via its additive generator g as $T(a,b)=g^{-1}(g(a)+g(b))$, where $g:[0,t]\to[0,+\infty]$ is a strictly decreasing function such that $g(t)=0$ and $g(0)=+\infty$, and a dual extended t-conorm $S(a,b)$ is expressed as $S(a,b)=f^{-1}(f(a)+f(b))$ with $f(x)=g(t-x)$.

Theorem 2.1. Suppose that $g:[0,t]\to[0,+\infty]$ is a strictly decreasing function such that $g(t)=0$ and $g(0)=+\infty$, $f(x)=g(t-x)$, $T(a,b)=g^{-1}(g(a)+g(b))$, and $S(a,b)=f^{-1}(f(a)+f(b))$. Then, $T(a,b)$ is an extended t-norm and $S(a,b)$ is a dual extended t-conorm.

Proof. Assume that $a,b,c,d\in[0,t]$, then

(1) $T(t,a)=g^{-1}(g(t)+g(a))=g^{-1}(g(a))=a$

$S(0,a)=f^{-1}(f(0)+f(a))=f^{-1}(f(a))=a$

(2) $T(a,b)=g^{-1}(g(a)+g(b))=g^{-1}(g(b)+g(a))=T(b,a)$

$S(a,b)=f^{-1}(f(a)+f(b))=f^{-1}(f(b)+f(a))=S(b,a)$

(3)

$$\begin{aligned}
T(a,T(b,c)) &= g^{-1}(g(a)+g(T(b,c))) \\
&= g^{-1}\left(g(a)+g\left(g^{-1}(g(b)+g(c))\right)\right) \\
&= g^{-1}(g(a)+g(b)+g(c)) \\
&= g^{-1}\left(g\left(g^{-1}(g(a)+g(b))\right)+g(c)\right) \\
&= g^{-1}(g(T(a,b))+g(c)) \\
&= T(T(a,b),c)
\end{aligned}$$

$$\begin{aligned}
S(a,S(b,c)) &= f^{-1}(f(a)+f(S(b,c))) \\
&= f^{-1}\left(f(a)+f\left(f^{-1}(f(b)+f(c))\right)\right) \\
&= f^{-1}(f(a)+f(b)+f(c)) \\
&= f^{-1}\left(f\left(f^{-1}(f(a)+f(b))\right)+f(c)\right) \\
&= f^{-1}(f(S(a,b))+g(c)) \\
&= S(S(a,b),c)
\end{aligned}$$

(4) Because $g:[0,t]\to[0,+\infty]$ is a strictly decreasing function such that $g(t)=0$ and $g(0)=+\infty$, $g^{-1}:[0,+\infty]\to[0,t]$ exists and is also a strictly decreasing function such that $g^{-1}(0)=t$ and $g^{-1}(+\infty)=0$. If $a\le c$ and $b\le d$, then $g(c)\le g(a)$ and $g(d)\le g(b)$, so $T(a,b)=g^{-1}\left(g(a)+g(b)\right)\le g^{-1}\left(g(c)+g(d)\right)=T(c,d)$

Because $f(x)=g(t-x)$, $f:[0,t]\to[0,+\infty]$ is a strictly increasing function such that $f(0)=0$ and $f(t)=+\infty$, and $f^{-1}:[0,+\infty]\to[0,t]$ exists and is also a strictly increasing function such that $f^{-1}(0)=0$ and $f^{-1}(+\infty)=t$. Thus, we have $S(a,b)=f^{-1}\left(f(a)+f(b)\right)\le f^{-1}\left(f(c)+f(d)\right)=S(c,d)$. □

If we assign specific forms to the function g, then some extended t-conorms and t-norms can be obtained:

(1) Let $g(x)=-\log\left(\frac{x}{t}\right)$, then $f(x)=-\log\left(1-\frac{x}{t}\right)$, $g^{-1}(x)=te^{-x}$, $f^{-1}(x)=t\left(1-e^{-x}\right)$, and the extended Algebraic t-conorm and t-norm are obtained as follows:

$$S_{EA}(a,b)=a+b-\frac{ab}{t},\qquad T_{EA}(a,b)=\frac{ab}{t} \tag{1}$$

(2) Let $g(x)=\log\left(\frac{2t-x}{x}\right)$, then $f(x)=\log\left(\frac{t+x}{t-x}\right)$, $g^{-1}(x)=\frac{2t}{1+e^x}$, $f^{-1}(x)=\frac{\left(e^x-1\right)t}{e^x+1}$, and the extended Einstein t-conorm and t-norm are obtained as follows:

$$S_{EE}(a,b)=\frac{t^2(a+b)}{t^2+ab},\qquad T_{EE}(a,b)=\frac{tab}{2t^2-t(a+b)+ab} \tag{2}$$

(3) Let $g(x)=\log\left(\frac{\theta t+(1-\theta)x}{x}\right)$, $\theta>0$, then $f(x)=\log\left(\frac{t+(\theta-1)x}{t-x}\right)$, $g^{-1}(x)=\frac{\theta t}{e^x+\theta-1}$, $f^{-1}(x)=\frac{t\left(e^x-1\right)}{e^x+\theta-1}$, and the extended Hamacher t-conorm and t-norm are obtained as follows:

$$S_{EH}(a,b)=\frac{t^2(a+b)+t(\theta-2)ab}{t^2+(\theta-1)ab},\; T_{EH}(a,b)=\frac{tab}{\theta t^2+(1-\theta)t(a+b)+ab(\theta-1)} \tag{3}$$

Especially, if $\theta=1$, then the extended Hamacher t-conorm and t-norm reduce to the extended Algebraic t-conorm and t-norm respectively; if $\theta=2$, then the extended Hamacher t-conorm and t-norm reduce to the extended Einstein t-conorm and t-norm respectively.

(4) Let $g(x)=\log\left(\frac{\theta-1}{\theta^{\frac{x}{t}}-1}\right)$, $\theta>1$, then $f(x)=\log\left(\frac{\theta-1}{\theta^{1-\frac{x}{t}}-1}\right)$, $g^{-1}(x)=t\log_{\theta}\left(\frac{\theta-1+e^{x}}{e^{x}}\right)$, $f^{-1}(x)=t-t\log_{\theta}\left(\frac{\theta-1+e^{x}}{e^{x}}\right)$, and the extended Frank t-conorm and t-norm are obtained as follows:

$$S_{EF}(a,b)=t-t\log_{\theta}\left(1+\frac{\left(\theta^{1-\frac{a}{t}}-1\right)\cdot\left(\theta^{1-\frac{b}{t}}-1\right)}{\theta-1}\right) \quad , \quad T_{EF}(a,b)=t\log_{\theta}\left(1+\frac{\left(\theta^{\frac{a}{t}}-1\right)\cdot\left(\theta^{\frac{b}{t}}-1\right)}{\theta-1}\right)$$

(4)

Especially, if $\theta\to1$, then we have

$$\lim_{\theta\to1}g(x)=\lim_{\theta\to1}\log\left(\frac{\theta-1}{\theta^{\frac{x}{t}}-1}\right)=\log\left(\lim_{\theta\to1}\frac{\theta-1}{\theta^{\frac{x}{t}}-1}\right)=\log\left(\lim_{\theta\to1}\frac{t}{x\theta^{\frac{x}{t}-1}}\right)=-\log\left(\frac{x}{t}\right)$$

which indicates that $\lim_{\theta\to1}S_{EF}(a,b)=S_{EA}(a,b)$ and $\lim_{\theta\to1}T_{EF}(a,b)=T_{EA}(a,b)$. Namely, if $\theta\to1$, then the extended Frank t-conorm and t-norm reduce to the extended Algebraic t-conorm and t-norm, respectively.

(5) Let $g(x)=\frac{1}{x}-\frac{1}{t}$, then $f(x)=\frac{x}{t(t-x)}$, $g^{-1}(x)=\frac{t}{1+tx}$, $f^{-1}(x)=\frac{xt^{2}}{1+xt}$, and the extended t-conorm and t-norm are as follows:

$$S(a,b)=\frac{t^{2}(a+b)-2tab}{t^{2}-ab}, \qquad T(a,b)=\frac{tab}{t(a+b)-ab} \tag{5}$$

(6) Let $g(x)=\exp\left(\frac{t-x}{x}\right)-1$, then $f(x)=\exp\left(\frac{x}{t-x}\right)-1$, $g^{-1}(x)=\frac{t}{1+\log(1+x)}$, $f^{-1}(x)=\frac{t\log(1+x)}{1+\log(1+x)}$, and the extended t-conorm and t-norm are as follows:

$$S(a,b)=\frac{t\log\left(\exp\left(\frac{a}{t-a}\right)+\exp\left(\frac{b}{t-b}\right)-1\right)}{1+\log\left(\exp\left(\frac{a}{t-a}\right)+\exp\left(\frac{b}{t-b}\right)-1\right)},\ T(a,b)=\frac{t}{1+\log\left(\exp\left(\frac{t-a}{a}\right)+\exp\left(\frac{t-b}{b}\right)-1\right)} \tag{6}$$

2.3 The OWA, IOWA, GOWA, and IGOWA Operators

Yager [34] provided a definition of the ordered weighted averaging (OWA) operator as follows:

Definition 2.8 [34]. An OWA operator of dimension n is a mapping $\mathrm{OWA}: R^n \to R$ defined by an associated weighting vector $w = (w_1, w_2, \cdots, w_n)^T$ of dimension n, such that $w_j \in [0,1]$, $j = 1, 2, \cdots, n$, and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$\mathrm{OWA}(a_1, a_2, \cdots, a_n) = \sum_{j=1}^{n} w_j b_j$$

where b_j is the *j*th largest of the a_i ($i = 1, 2, \cdots, n$).

An important feature of OWA operator is the reordering step, which makes this a nonlinear operator. During this step the arguments are ordered by their values.

Yager and Filev [35] introduced the induced ordered weighted averaging (IOWA) operator as an extension of the OWA operator. Different from the OWA operator, the reordering step of the IOWA is carried out with order-inducing variables, rather than depending on the values of the arguments a_i. The IOWA operator can be defined as follows:

Definition 2.9 [35]. An IOWA is defined as follows:

$$\mathrm{IOWA}(\langle u_1, a_1 \rangle, \langle u_2, a_2 \rangle, \cdots, \langle u_n, a_n \rangle) = \sum_{j=1}^{n} w_j b_j$$

where $w = (w_1, w_2, \cdots, w_n)^T$ is an associated weighting vector, such that $w_j \in [0,1]$, $j = 1, 2, \cdots, n$, $\sum_{j=1}^{n} w_j = 1$, b_j is the a_i value of the OWA pair $\langle u_i, a_i \rangle$ having the *j*th largest u_i, and u_i in $\langle u_i, a_i \rangle$ is referred to as the order inducing variable and a_i as the argument variable.

The generalized OWA (GOWA) operator was introduced in [36]. It uses generalized means in the OWA operator. It can be defined as follows:

Definition 2.10 [36]. A GOWA operator of dimension n is a mapping $\mathrm{GOWA}: R^n \to R$, according to the following formula:

$$\mathrm{GOWA}(a_1, a_2, \cdots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda}$$

where $\lambda \in (-\infty, +\infty)$, $w = (w_1, w_2, \cdots, w_n)^T$ is the weighting vector of $(a_1, a_2, \cdots, a_n)$, $w_j \in [0,1]$, $j = 1, 2, \cdots, n$, and $\sum_{j=1}^{n} w_j = 1$, b_j is the *j*th largest of a_i ($i = 1, 2, \cdots, n$).

Merigo and Gil-Lafuente [37] proposed a generalization of OWA operator by using generalized means and order inducing variables called the induced generalized ordered weighted averaging (IGOWA) operator, which is defined as follows:

Definition 2.11 [37]. An IGOWA operator of dimension n is a mapping $\mathrm{IGOWA}: R^n \rightarrow R$ defined by an associated weighting vector $w = \left(w_1, w_2, \cdots, w_n\right)^T$ of dimension n, such that $w_j \in \left[0,1\right]$, $j = 1,2,\cdots,n$, and $\sum_{j=1}^{n} w_j = 1$, according to the following expression:

$$\mathrm{IGOWA}\left(\left\langle u_1, a_1\right\rangle, \left\langle u_2, a_2\right\rangle, \cdots, \left\langle u_n, a_n\right\rangle\right) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda}$$

where $\lambda \in \left(-\infty, +\infty\right)$, b_j is the a_j value of the OWA pair $\left\langle u_j, a_j\right\rangle$ having the *j*th largest u_j, and u_j in $\left\langle u_j, a_j\right\rangle$ is referred to as the order inducing variable and a_j as the argument variable.

The IGOWA operators, however, can only be used in situations where the aggregated arguments are the exact numerical values. In the following we shall investigate an induced generalized uncertain linguistic OWA operator, which can be used in situations where the aggregated arguments are given in the form of uncertain linguistic variables.

3. INDUCED GENERALIZED UNCERTAIN LINGUISTIC AGGREGATION OPERATORS

In this section, some new operational laws in the linguistic and uncertain linguistic environment are proposed, and some properties of the new laws are studied. Moreover, based on these new laws, we develop several new uncertain linguistic aggregation operators and investigate some desired properties of the developed operators.

3.1 Some New Operational Laws for Linguistic Variables and Uncertain Linguistic Variables

In the following, based on the extended t-conorm and t-norm, we define some new operational laws for linguistic variables and uncertain linguistic variables.

Definition 3.1. For $s_\alpha, s_\beta \in \overline{S}$, and $\lambda \geq 0$, we define some new operational laws as follows:

(1) $s_\alpha \oplus s_\beta = s_{S(\alpha,\beta)} = s_{f^{-1}\left(f(\alpha)+f(\beta)\right)}$;

(2) $\lambda s_\alpha = s_{f^{-1}\left(\lambda f(\alpha)\right)}$;

(3) $s_\alpha \otimes s_\beta = s_{T(\alpha,\beta)} = s_{g^{-1}\left(g(\alpha)+g(\beta)\right)}$;

(4) $s_\alpha^\lambda = s_{g^{-1}\left(\lambda g(\alpha)\right)}$.

Especially, if $g(x) = -\log\left(\frac{x}{t}\right)$, then we have:

(5) $s_\alpha \oplus s_\beta = s_{\alpha+\beta-\frac{\alpha\beta}{t}}$; (6) $\lambda s_\alpha = s_{t\cdot\left[1-\left(1-\frac{\alpha}{t}\right)^\lambda\right]}$; (7) $s_\alpha \otimes s_\beta = s_{\frac{\alpha\beta}{t}}$; (8) $s_\alpha^\lambda = s_{t\cdot\left(\frac{\alpha}{t}\right)^\lambda}$

If $g(x) = \log\left(\frac{2t-x}{x}\right)$, then we have:

(9) $s_\alpha \oplus s_\beta = s_{\frac{t^2(\alpha+\beta)}{t^2+\alpha\beta}}$; (10) $\lambda s_\alpha = s_{t\cdot\frac{\left(\frac{t+\alpha}{t-\alpha}\right)^\lambda-1}{\left(\frac{t+\alpha}{t-\alpha}\right)^\lambda+1}}$; (11) $s_\alpha \otimes s_\beta = s_{\frac{t\alpha\beta}{2t^2-t(\alpha+\beta)+\alpha\beta}}$; (12)

$$s_\alpha^\lambda = s_{\frac{2t\alpha^\lambda}{\alpha^\lambda+(2t-\alpha)^\lambda}}.$$

If $g(x) = \log\left(\frac{\theta t+(1-\theta)x}{x}\right)$, $\theta > 0$, then we have:

(13) $s_\alpha \oplus s_\beta = s_{\frac{t^2(\alpha+\beta)+t(\theta-2)\alpha\beta}{t^2+(\theta-1)\alpha\beta}}$; (14) $\lambda s_\alpha = s_{\frac{t\left[\left(\frac{t+(\theta-1)\alpha}{t-\alpha}\right)^\lambda-1\right]}{\left(\frac{t+(\theta-1)\alpha}{t-\alpha}\right)^\lambda+\theta-1}}$; (15) $s_\alpha \otimes s_\beta = s_{\frac{t\alpha\beta}{\theta t^2+(1-\theta)t(\alpha+\beta)+\alpha\beta(\theta-1)}}$;

(16) $s_\alpha^\lambda = s_{\frac{\theta t}{\left(\frac{\theta t+(1-\theta)\alpha}{\alpha}\right)^\lambda+\theta-1}}$.

Especially, if $\theta = 1$, then (13)–(16) reduce to (5)–(8); if $\theta = 2$, then (13)–(16) reduce to (9)–(12)..

If $g(x) = \log\left(\frac{\theta-1}{\theta^{\frac{x}{t}}-1}\right)$, $\theta > 1$, then we have:

(17) $s_\alpha \oplus s_\beta = s_{t-t\log_\theta\left(1+\frac{\left(\theta^{1-\frac{\alpha}{t}}-1\right)\cdot\left(\theta^{1-\frac{\beta}{t}}-1\right)}{\theta-1}\right)}$; (18) $\lambda s_\alpha = s_{t-t\cdot\log_\theta\left[\frac{\theta-1+\left(\frac{\theta-1}{\theta^{t-\alpha}-1}\right)^\lambda}{\left(\frac{\theta-1}{\theta^{t-\alpha}-1}\right)^\lambda}\right]}$; (19)

$s_\alpha \otimes s_\beta = s_{t\log_\theta\left(1+\frac{\left(\theta^{\frac{\alpha}{t}}-1\right)\cdot\left(\theta^{\frac{\beta}{t}}-1\right)}{\theta-1}\right)}$;

(20) $s_\alpha^\lambda = s_{t\cdot\log_\theta\left[\frac{\theta-1+\left(\frac{\theta-1}{\theta^\alpha-1}\right)^\lambda}{\left(\frac{\theta-1}{\theta^\alpha-1}\right)^\lambda}\right]}$.

Especially, if $\theta \to 1$, then (17)–(20) reduce to (5)–(8).

If $g(x)=\frac{1}{x}-\frac{1}{t}$, then we have:

(21) $s_\alpha \oplus s_\beta = s_{\frac{t^2(\alpha+\beta)-2t\alpha\beta}{t^2-\alpha\beta}}$; (22) $\lambda s_\alpha = s_{\frac{\lambda t\alpha}{t+(\lambda-1)\alpha}}$; (23) $s_\alpha \otimes s_\beta = s_{\frac{t\alpha\beta}{t(\alpha+\beta)-\alpha\beta}}$; (24) $s_\alpha^\lambda = s_{\frac{t\alpha}{\alpha+\lambda(t-\alpha)}}$.

If $g(x)=\exp\left(\frac{t-x}{x}\right)-1$, then we have:

(25) $s_\alpha \oplus s_\beta = s_{\frac{t\log\left(\exp\left(\frac{\alpha}{t-\alpha}\right)+\exp\left(\frac{\beta}{t-\beta}\right)-1\right)}{1+\log\left(\exp\left(\frac{\alpha}{t-\alpha}\right)+\exp\left(\frac{\beta}{t-\beta}\right)-1\right)}}$; (26) $\lambda s_\alpha = s_{\frac{t\log\left(\lambda\exp\left(\frac{\alpha}{t-\alpha}\right)+1-\lambda\right)}{1+\log\left(\lambda\exp\left(\frac{\alpha}{t-\alpha}\right)+1-\lambda\right)}}$;

(27) $s_\alpha \otimes s_\beta = s_{\frac{t}{1+\log\left(\exp\left(\frac{t-\alpha}{\alpha}\right)+\exp\left(\frac{t-\beta}{\beta}\right)-1\right)}}$; (28) $s_\alpha^\lambda = s_{\frac{t}{1+\log\left(\lambda\exp\left(\frac{t-\alpha}{\alpha}\right)+1-\lambda\right)}}$.

Theorem 3.1. For $s_\alpha, s_\beta, s_\gamma, s_\delta \in \overline{S}$, and $\lambda > 0$, we have the following properties:

(1) $\max\{s_\alpha, s_\beta\} \le s_\alpha \oplus s_\beta \le s_t$.

$s_\alpha \oplus s_\beta = s_t$ if and only if at least one of s_α and s_β is equal to s_t.

$s_\alpha \oplus s_\beta = s_0$ if and only if $s_\alpha = s_\beta = s_0$.

(2) $s_0 \le s_\alpha \otimes s_\beta \le \min\{s_\alpha, s_\beta\}$.

$s_\alpha \otimes s_\beta = s_0$ if and only if at least one of s_α and s_β is equal to s_0.

$s_\alpha \otimes s_\beta = s_t$ if and only if $s_\alpha = s_\beta = s_t$.

(3) $s_0 \le \lambda s_\alpha \le s_t$. $\lambda s_\alpha = s_0$ if and only if $s_\alpha = s_0$. $\lambda s_\alpha = s_t$ if and only if $s_\alpha = s_t$.

(4) $s_0 \le s_\alpha^\lambda \le s_t$. $s_\alpha^\lambda = s_0$ if and only if $s_\alpha = s_0$. $s_\alpha^\lambda = s_t$ if and only if $s_\alpha = s_t$.

(5) $s_\alpha \oplus s_\beta \le s_\gamma \oplus s_\delta$, if $\alpha \le \gamma$ and $\beta \le \delta$; Furthermore, $s_\alpha \oplus s_\beta = s_\gamma \oplus s_\delta$ if and only if

(i) $s_\alpha = s_\gamma$ and $s_\beta = s_\delta$, or

(ii) at least one of s_α and s_β is equal to s_t, and at least one of s_γ and s_δ is equal to s_t.

(6) $s_\alpha \otimes s_\beta \le s_\gamma \otimes s_\delta$, if $\alpha \le \gamma$ and $\beta \le \delta$; Furthermore, $s_\alpha \otimes s_\beta = s_\gamma \otimes s_\delta$ if and only

(i) $s_\alpha = s_\gamma$ and $s_\beta = s_\delta$, or

(ii) at least one of s_α and s_β is equal to s_0, and at least one of s_γ and s_δ is equal to s_0.

Proof: (1) According to the definition of f , $f^{-1}:[0,+\infty]\to[0,t]$ exists and is also a strictly monotonically increasing and continuous function such that $f^{-1}(0)=0$ and $f^{-1}(+\infty)=t$. So,

$$\alpha=f^{-1}(f(\alpha))\le f^{-1}(f(\alpha)+f(\beta))\le f^{-1}(f(t)+f(t))=f^{-1}(+\infty)=t$$

and

$$\beta=f^{-1}(f(\beta))\le f^{-1}(f(\alpha)+f(\beta))\le f^{-1}(f(t)+f(t))=f^{-1}(+\infty)=t$$

Thus, we have

$$\max\{\alpha,\beta\}\le f^{-1}(f(\alpha)+f(\beta))\le t$$

which implies

$$\max\{s_\alpha,s_\beta\}\le s_\alpha\oplus s_\beta\le s_t$$

Furthermore, if $s_\alpha\oplus s_\beta=s_t$, then $f^{-1}(f(\alpha)+f(\beta))=t$, which means that $f(\alpha)+f(\beta)=+\infty$. Therefore, we have $f(\alpha)=+\infty$ or $f(\beta)=+\infty$, which implies that $\alpha=t$ or $\beta=t$, i.e., at least one of s_α and s_β is equal to s_t.

On the other hand, if at least one of s_α and s_β is equal to s_t, then $\alpha=t$ or $\beta=t$, which means that $f(\alpha)+f(\beta)=+\infty$ and $f^{-1}(f(\alpha)+f(\beta))=t$, Thus, we have

$$s_\alpha\oplus s_\beta=s_{f^{-1}(f(\alpha)+f(\beta))}=s_t .$$

If $s_\alpha\oplus s_\beta=s_0$, then $f^{-1}(f(\alpha)+f(\beta))=0$, which means that $f(\alpha)+f(\beta)=0$. Therefore, we have $f(\alpha)=0$ and $f(\beta)=0$, which implies that $s_\alpha=s_0$ and $s_\beta=s_0$.

On the other hand, if $s_\alpha=s_0$ and $s_\beta=s_0$, then $\alpha=\beta=0$, which means $f(\alpha)+f(\beta)=0$ and $f^{-1}(f(\alpha)+f(\beta))=0$, Thus, we have

$$s_\alpha\oplus s_\beta=s_{f^{-1}(f(\alpha)+f(\beta))}=s_0 .$$

(2) According to the definition of g , $g^{-1}:[0,+\infty]\to[0,t]$ exists and is also a strictly monotonically decreasing and continuous function such that $g^{-1}(+\infty)=0$ and $g^{-1}(0)=t$. So,

$$0=g^{-1}(+\infty)=g^{-1}(g(0)+g(0))\le g^{-1}(g(\alpha)+g(\beta))\le g^{-1}(g(\alpha))=\alpha$$

and

$$0=g^{-1}(+\infty)=g^{-1}(g(0)+g(0))\le g^{-1}(g(\alpha)+g(\beta))\le g^{-1}(g(\beta))=\beta$$

Thus, we have

$$0\le g^{-1}(g(\alpha)+g(\beta))\le\min\{\alpha,\beta\}$$

which implies

$$s_0\le s_\alpha\otimes s_\beta\le\min\{s_\alpha,s_\beta\}$$

Furthermore, if $s_\alpha \otimes s_\beta = s_0$, then $g^{-1}\left(g(\alpha)+g(\beta)\right)=0$, which means that $g(\alpha)+g(\beta)=+\infty$. Therefore, we have $g(\alpha)=+\infty$ or $g(\beta)=+\infty$, which implies that $\alpha=0$ or $\beta=0$, i.e., at least one of s_α and s_β is equal to s_0.

On the other hand, if at least one of s_α and s_β is equal to s_0, then $\alpha=0$ or $\beta=0$, which implies that $g(\alpha)+g(\beta)=+\infty$ and $g^{-1}\left(g(\alpha)+g(\beta)\right)=0$, Thus, we have

$s_\alpha \otimes s_\beta = s_{g^{-1}(g(\alpha)+g(\beta))} = s_0$.

If $s_\alpha \otimes s_\beta = s_t$, then $g^{-1}\left(g(\alpha)+g(\beta)\right)=t$, which means that $g(\alpha)+g(\beta)=0$. Therefore, we have $g(\alpha)=g(\beta)=0$, which implies that $\alpha=\beta=t$, i.e., $s_\alpha=s_\beta=s_t$.

On the other hand, if $s_\alpha=s_\beta=s_t$, then $\alpha=\beta=t$, which implies $g(\alpha)+g(\beta)=0$ and $g^{-1}\left(g(\alpha)+g(\beta)\right)=t$, Thus, we have

$s_\alpha \otimes s_\beta = s_{g^{-1}(g(\alpha)+g(\beta))} = s_t$.

(3) Since $0=f^{-1}\left(\lambda f(0)\right)\le f^{-1}\left(\lambda f(\alpha)\right)\le f^{-1}\left(\lambda f(t)\right)=t$, then $s_0 \le \lambda s_\alpha \le s_t$.

If $\lambda s_\alpha = s_0$, then $f^{-1}\left(\lambda f(\alpha)\right)=0$, which means that $f(\alpha)=0$ and $s_\alpha=s_0$. Conversely, if $s_\alpha=s_0$, then $\alpha=0$ and $f^{-1}\left(\lambda f(\alpha)\right)=0$, which imply that $\lambda s_\alpha = s_{f^{-1}(\lambda f(\alpha))}=s_0$.

If $\lambda s_\alpha = s_t$, then $f^{-1}\left(\lambda f(\alpha)\right)=t$, which means that $f(\alpha)=+\infty$ and $s_\alpha=s_t$. Conversely, if $s_\alpha=s_t$, then $\alpha=t$ and $f^{-1}\left(\lambda f(\alpha)\right)=t$, which imply that $\lambda s_\alpha = s_{f^{-1}(\lambda f(\alpha))}=s_t$.

(4) Since $0=g^{-1}\left(\lambda g(0)\right)\le g^{-1}\left(\lambda g(\alpha)\right)\le g^{-1}\left(\lambda g(t)\right)=t$, then $s_0 \le s_\alpha^\lambda \le s_t$.

If $s_\alpha^\lambda = s_0$, then $g^{-1}\left(\lambda g(\alpha)\right)=0$, which means that $g(\alpha)=+\infty$ and $s_\alpha=s_0$. Conversely, if $s_\alpha=s_0$, then $\alpha=0$ and $g^{-1}\left(\lambda g(\alpha)\right)=0$, which imply that $s_\alpha^\lambda = s_{g^{-1}(\lambda g(\alpha))}=s_0$.

If $s_\alpha^\lambda = s_t$, then $g^{-1}\left(\lambda g(\alpha)\right)=t$, which means that $g(\alpha)=0$ and $s_\alpha=s_t$. Conversely, if $s_\alpha=s_t$, then $\alpha=t$ and $g^{-1}\left(\lambda g(\alpha)\right)=t$, which imply that $s_\alpha^\lambda = s_{g^{-1}(\lambda g(\alpha))}=s_t$.

(5) If $\alpha\le\gamma$ and $\beta\le\delta$, then $f(\alpha)\le f(\gamma)$, $f(\beta)\le f(\delta)$ and $f^{-1}\left(f(\alpha)+f(\beta)\right)\le f^{-1}\left(f(\gamma)+f(\delta)\right)$. Thus, we have $s_\alpha \oplus s_\beta = s_{f^{-1}(f(\alpha)+f(\beta))} \le s_{f^{-1}(f(\gamma)+f(\delta))} = s_\gamma \oplus s_\delta$.

If $s_\alpha \oplus s_\beta = s_\gamma \oplus s_\delta$, then we consider the following two cases:

Case 1: $s_\alpha \oplus s_\beta = s_\gamma \oplus s_\delta = s_t$. In this case, according to Theorem 3.1 (1), at least one of s_α and s_β is equal to s_t, and at least one of s_γ and s_δ is equal to s_t, and vice versa.

Case 2: $s_\alpha \oplus s_\beta = s_\gamma \oplus s_\delta < s_t$. In this case, according to Definition 3.1, we have $f^{-1}\left(f(\alpha)+f(\beta)\right)=f^{-1}\left(f(\gamma)+f(\delta)\right)$ and $f(\alpha)+f(\beta)=f(\gamma)+f(\delta)$. Since $f(\alpha)\le f(\gamma)$ and $f(\beta)\le f(\delta)$, then $f(\alpha)=f(\gamma)$ and $f(\beta)=f(\delta)$, which implies that $\alpha=\gamma$ and $\beta=\delta$, i.e., $s_\alpha = s_\gamma$ and $s_\beta = s_\delta$, and vice versa.

(6) If $\alpha\le\gamma$ and $\beta\le\delta$, then $g(\alpha)\ge g(\gamma)$, $g(\beta)\ge g(\delta)$ and $g^{-1}\left(g(\alpha)+g(\beta)\right)\le g^{-1}\left(g(\gamma)+g(\delta)\right)$. Thus, we have $s_\alpha \otimes s_\beta = s_{g^{-1}(g(\alpha)+g(\beta))} \le s_{g^{-1}(g(\gamma)+g(\delta))} = s_\gamma \oplus s_\delta$.

If $s_\alpha \otimes s_\beta = s_\gamma \otimes s_\delta$, then we consider the following two cases:

Case 1: $s_\alpha \otimes s_\beta = s_\gamma \otimes s_\delta = s_0$. In this case, according to Theorem 3.1 (2), at least one of s_α and s_β is equal to s_0, and at least one of s_γ and s_δ is equal to s_0, and vice versa.

Case 2: $s_\alpha \otimes s_\beta = s_\gamma \otimes s_\delta > s_0$. In this case, according to Definition 3.1, we have $g^{-1}\left(g(\alpha)+g(\beta)\right)=g^{-1}\left(g(\gamma)+g(\delta)\right)$ and $g(\alpha)+g(\beta)=g(\gamma)+g(\delta)$. Since $g(\alpha)\ge g(\gamma)$ and $g(\beta)\ge g(\delta)$, then $g(\alpha)=g(\gamma)$ and $g(\beta)=g(\delta)$, which implies that $\alpha=\gamma$ and $\beta=\delta$, i.e., $s_\alpha = s_\gamma$ and $s_\beta = s_\delta$, and vice versa.

Theorem 3.2. For $s_\alpha, s_\beta \in \bar{S}$, and $\lambda,\lambda_1,\lambda_2>0$, we have the following properties:

(1) $s_\alpha \oplus s_\beta = s_\beta \oplus s_\alpha$; (2) $\left(s_\alpha \oplus s_\beta\right)\oplus s_\gamma = s_\alpha \oplus\left(s_\beta \oplus s_\gamma\right)$; (3) $s_0 \oplus s_\alpha = s_\alpha$; (4) $\lambda\left(s_\alpha \oplus s_\beta\right)=\lambda s_\alpha \oplus \lambda s_\beta$; (5) $\lambda_1 s_\alpha \oplus \lambda_2 s_\alpha = \left(\lambda_1+\lambda_2\right)s_\alpha$; (6) $s_\alpha \otimes s_\beta = s_\beta \otimes s_\alpha$; (7) $\left(s_\alpha \otimes s_\beta\right)\otimes s_\gamma = s_\alpha \otimes\left(s_\beta \otimes s_\gamma\right)$; (8) $s_t \otimes s_\alpha = s_\alpha$; (9) $\left(s_\alpha \otimes s_\beta\right)^\lambda = s_\alpha^\lambda \otimes s_\beta^\lambda$; (10) $s_\alpha^{\lambda_1} \otimes s_\alpha^{\lambda_2} = s_\alpha^{\lambda_1+\lambda_2}$.

Proof: (1) $s_\alpha \oplus s_\beta = s_{f^{-1}(f(\alpha)+f(\beta))} = s_{f^{-1}(f(\beta)+f(\alpha))} = s_\beta \oplus s_\alpha$

(2) $\left(s_\alpha \oplus s_\beta\right)\oplus s_\gamma = \left(s_{f^{-1}(f(\alpha)+f(\beta))}\right)\oplus s_\gamma = s_{f^{-1}\left(f\left(f^{-1}(f(\alpha)+f(\beta))\right)+f(\gamma)\right)} = s_{f^{-1}(f(\alpha)+f(\beta)+f(\gamma))}$

$s_\alpha \oplus\left(s_\beta \oplus s_\gamma\right) = s_\alpha \oplus\left(s_{f^{-1}(f(\beta)+f(\gamma))}\right) = s_{f^{-1}\left(f(\alpha)+f\left(f^{-1}(f(\beta)+f(\gamma))\right)\right)} = s_{f^{-1}(f(\alpha)+f(\beta)+f(\gamma))}$

(3) $s_0 \oplus s_\alpha = s_{f^{-1}(f(0)+f(\alpha))} = s_{f^{-1}(f(\alpha))} = s_\alpha$

(4) $\lambda\left(s_\alpha \oplus s_\beta\right) = \lambda s_{f^{-1}(f(\alpha)+f(\beta))} = s_{f^{-1}\left(\lambda f\left(f^{-1}(f(\alpha)+f(\beta))\right)\right)} = s_{f^{-1}(\lambda f(\alpha)+\lambda f(\beta))}$

$\lambda s_\alpha \oplus \lambda s_\beta = s_{f^{-1}(\lambda f(\alpha))} \oplus s_{f^{-1}(\lambda f(\beta))} = s_{f^{-1}\left(f\left(f^{-1}(\lambda f(\alpha))\right)+f\left(f^{-1}(\lambda f(\beta))\right)\right)} = s_{f^{-1}(\lambda f(\alpha)+\lambda f(\beta))}$

(5) $\lambda_1 s_\alpha \oplus \lambda_2 s_\alpha = s_{f^{-1}(\lambda_1 f(\alpha))} \oplus s_{f^{-1}(\lambda_2 f(\alpha))} = s_{f^{-1}\left(f\left(f^{-1}(\lambda_1 f(\alpha))\right)+f\left(f^{-1}(\lambda_2 f(\alpha))\right)\right)} = s_{f^{-1}(\lambda_1 f(\alpha)+\lambda_2 f(\alpha))}$

$\left(\lambda_1+\lambda_2\right)s_\alpha = s_{f^{-1}((\lambda_1+\lambda_2)f(\alpha))} = s_{f^{-1}(\lambda_1 f(\alpha)+\lambda_2 f(\alpha))}$

(6) $s_\alpha \otimes s_\beta = s_{g^{-1}(g(\alpha)+g(\beta))} = s_{g^{-1}(g(\beta)+g(\alpha))} = s_\beta \otimes s_\alpha$

(7) $\left(s_\alpha \otimes s_\beta\right) \otimes s_\gamma = \left(s_{g^{-1}(g(\alpha)+g(\beta))}\right) \oplus s_\gamma = s_{g^{-1}\left(g\left(g^{-1}(g(\alpha)+g(\beta))\right)+g(\gamma)\right)} = s_{g^{-1}(g(\alpha)+g(\beta)+g(\gamma))}$

$s_\alpha \otimes \left(s_\beta \otimes s_\gamma\right) = s_\alpha \oplus \left(s_{g^{-1}(g(\beta)+g(\gamma))}\right) = s_{g^{-1}\left(g(\alpha)+g\left(g^{-1}(g(\beta)+g(\gamma))\right)\right)} = s_{g^{-1}(g(\alpha)+g(\beta)+g(\gamma))}$

(8) $s_t \otimes s_\alpha = s_{g^{-1}(g(t)+g(\alpha))} = s_{g^{-1}(g(\alpha))} = s_\alpha$

(9) $\left(s_\alpha \otimes s_\beta\right)^\lambda = s^\lambda_{g^{-1}(g(\alpha)+g(\beta))} = s_{g^{-1}\left(\lambda g\left(g^{-1}(g(\alpha)+g(\beta))\right)\right)} = s_{g^{-1}(\lambda g(\alpha)+\lambda g(\beta))}$

$s_\alpha^\lambda \otimes s_\beta^\lambda = s_{g^{-1}(\lambda g(\alpha))} \otimes s_{g^{-1}(\lambda g(\beta))} = s_{g^{-1}\left(g\left(g^{-1}(\lambda g(\alpha))\right)+g\left(g^{-1}(\lambda g(\beta))\right)\right)} = s_{g^{-1}(\lambda g(\alpha)+\lambda g(\beta))}$

(10) $s_\alpha^{\lambda_1} \otimes s_\alpha^{\lambda_2} = s_{g^{-1}(\lambda_1 g(\alpha))} \otimes s_{g^{-1}(\lambda_2 g(\alpha))} = s_{g^{-1}\left(g\left(g^{-1}(\lambda_1 g(\alpha))\right)+g\left(g^{-1}(\lambda_2 g(\alpha))\right)\right)} = s_{g^{-1}(\lambda_1 g(\alpha)+\lambda_2 g(\alpha))}$

$s_\alpha^{\lambda_1+\lambda_2} = s_{g^{-1}((\lambda_1+\lambda_2)g(\alpha))} = s_{g^{-1}(\lambda_1 g(\alpha)+\lambda_2 g(\alpha))}$

This completes the proof. □

Theorem 3.3. Let $s_\alpha, s_\beta \in \overline{S}$ and $\lambda > 0$, then the following are also valid:

(1) $\left(\mathrm{neg}(s_\alpha)\right)^\lambda = \mathrm{neg}(\lambda s_\alpha)$. (2) $\lambda \mathrm{neg}(s_\alpha) = \mathrm{neg}\left(s_\alpha^\lambda\right)$.

(3) $\mathrm{neg}(s_\alpha) \oplus \mathrm{neg}(s_\beta) = \mathrm{neg}(s_\alpha \otimes s_\beta)$. (4) $\mathrm{neg}(s_\alpha) \otimes \mathrm{neg}(s_\beta) = \mathrm{neg}(s_\alpha \oplus s_\beta)$.

Proof. According to the operations defined in Definition 3.1, we have

(1) $\left(\mathrm{neg}(s_\alpha)\right)^\lambda = s^\lambda_{t-\alpha} = s_{g^{-1}(\lambda g(t-\alpha))} = s_{g^{-1}(\lambda f(\alpha))} = s_{t-f^{-1}(\lambda f(\alpha))} = \mathrm{neg}(\lambda s_\alpha)$

(2) $\lambda \mathrm{neg}(s_\alpha) = \lambda s_{t-\alpha} = s_{f^{-1}(\lambda f(t-\alpha))} = s_{f^{-1}(\lambda g(\alpha))} = s_{t-g^{-1}(\lambda g(\alpha))} = \mathrm{neg}\left(s_\alpha^\lambda\right)$

(3) $\mathrm{neg}(s_\alpha) \oplus \mathrm{neg}(s_\beta) = s_{t-\alpha} \oplus s_{t-\beta} = s_{f^{-1}(f(t-\alpha)+f(t-\beta))} = s_{f^{-1}(g(\alpha)+g(\beta))} = s_{t-g^{-1}(g(\alpha)+g(\beta))} = \mathrm{neg}(s_\alpha \otimes s_\beta)$

(4) $\mathrm{neg}(s_\alpha) \otimes \mathrm{neg}(s_\beta) = s_{t-\alpha} \otimes s_{t-\beta} = s_{g^{-1}(g(t-\alpha)+g(t-\beta))} = s_{g^{-1}(f(\alpha)+f(\beta))} = s_{t-f^{-1}(f(\alpha)+f(\beta))} = \mathrm{neg}(s_\alpha \oplus s_\beta)$

This completes the proof.

Remark 3.1. In the following, we will compare the new operational laws with the Xu' operational laws [5,7] and then illustrate the advantages of the new operational laws over the Xu's operational laws.

(1) The Xu' operational laws only perform a simple aggregation on the linguistic terms. While the new operational laws are defined on the basis of the extended t-conorm and t-norm, which are generated by an additive function $g(x)$ and its dual function $f(x) = g(t-x)$. When the additive generator $g(t)$ is assigned different forms, we can obtain some specific extended t-conorms and t-norms, and then obtain some specific operations on linguistic variables. The prominent characteristic of the developed operational laws is that they include a variety of operations on linguistic variables when the additive generator g is assigned different forms.

(2) According to Definition 2.1, there are no relationships between the addition and multiplication operators proposed by Xu. Concretely, $s_\alpha \oplus_{Xu} s_\beta$ may be greater than $s \otimes_{Xu} s_\beta$ and it may also be less than $s \otimes_{Xu} s_\beta$. According to Theorem 3.1, we can obtain

that the $s_\alpha \oplus s_\beta$ is greater than or equal to the maximum value between s_α and s_β, while $s \otimes s_\beta$ is less than or equal to the minimum value between s_α and s_β, i.e., $s_0 \le s_\alpha \otimes s_\beta \le \min\{s_\alpha, s_\beta\} \le \max\{s_\alpha, s_\beta\} \le s_\alpha \oplus s_\beta \le s_t$. Therefore, the addition operation can obtain more favorable (or optimistic) expectations, and can therefore be referred to as an optimistic operation, while the multiplication operation produces more unfavorable (or pessimistic) expectations, and can therefore be referred to as a pessimistic operation.

(3) According to Definition 2.1, the position index of the linguistic term obtained with the Xu's addition operation between two linguistic terms s_α and s_β is the addition between the position index of s_α and the position index of s_β, i.e., $s_\alpha \oplus s_\beta = s_{\alpha+\beta}$. Similarly, the position index of the linguistic term obtained with the Xu's multiplication operation between two linguistic terms s_α and s_β is the multiplication between the position index of s_α and the position index of s_β, i.e., $s_\alpha \otimes s_\beta = s_{\alpha\times\beta}$. Therefore, the Xu's addition and multiplication operations can be referred to as two most optimistic linguistic operations and they do not fully consider the relationship and influence between two linguistic terms s_α and s_β. In contrast, the developed addition and multiplication operations overcome this drawback and fully consider the relationship and influence between two linguistic terms s_α and s_β. If there is a positive influence between s_α and s_β, then we use the developed addition operation $s_\alpha \oplus s_\beta$ and choose different function for various situations. If there is a passive influence between s_α and s_β, then we use the developed addition operation $s_\alpha \otimes s_\beta$ and choose different function for various situations.

(4) According to Definition 2.1, we have the following result: assume that $s_\alpha, s_\beta, s_\gamma, s_\delta \in \overline{S}$, $\alpha < \gamma$, and $\beta < \delta$. If $\alpha + \beta = t$, then $s_\alpha \oplus s_\beta = s_\gamma \oplus s_\delta = s_t$; if $\alpha \times \beta = t$, then $s_\alpha \otimes s_\beta = s_\gamma \otimes s_\delta = s_t$. That is, the Xu's addition and multiplication operations are not strictly monotonous, which is inconsistent with our intuitions. In contrast, according to Theorem 3.1 (1) and (2), if $s_\alpha, s_\beta, s_\gamma, s_\delta \in \overline{S}$, $0 < \alpha < \gamma < t$, and $0 < \beta < \delta < t$, then we have $s_\alpha \oplus s_\beta < s_\gamma \oplus s_\delta$ and $s_\alpha \otimes s_\beta < s_\gamma \otimes s_\delta$. Therefore, the developed addition and multiplication operations are strictly monotonous, which is consistent with our intuitions. An illustrative example is given as follows:

Example 3.1. Let s_3, s_4, s_5, and s_6 be four linguistic terms. Clearly, $s_3 < s_4$ and $s_5 < s_6$. Let $g(x) = \log\left(\dfrac{0.5\times 8 + (1-0.5)x}{x}\right)$, then $f(x) = \log\left(\dfrac{8 + (0.5-1)x}{8-x}\right)$, $g^{-1}(x) = \dfrac{0.5\times 8}{e^x + 0.5 - 1}$, $f^{-1}(x) = \dfrac{8\times(e^x - 1)}{e^x + 0.5 - 1}$. By Definition 3.1 (13) and (15), we have

$$s_3 \oplus s_5 = s_{f^{-1}(f(3)+f(5))} = s_{5.8761} \qquad s_4 \oplus s_6 = s_{f^{-1}(f(4)+f(6))} = s_{6.7692}$$

$$s_3 \otimes s_5 = s_{g^{-1}(g(3)+g(5))} = s_{2.1239} \qquad s_4 \otimes s_6 = s_{g^{-1}(g(4)+g(6))} = s_{3.2000}$$

It is obvious that $s_3 \oplus s_5 < s_4 \oplus s_6$ and $s_3 \otimes s_5 < s_4 \otimes s_6$, which is consistent with our intuition and can be more easily accepted.

If we use the operations $\oplus_{Xu}$ and $\otimes_{Xu}$ proposed by Xu [5,7] (Definition 2.1) to aggregate s_3 and s_5, and s_4 and s_6, respectively, then we have

$$s_3 \oplus_{Xu} s_5 = s_{3+5} = s_8 \qquad s_4 \oplus_{Xu} s_6 = s_{4+6} = s_8$$
$$s_3 \otimes_{Xu} s_5 = s_{3\times 5} = s_8 \qquad s_4 \otimes_{Xu} s_6 = s_{4\times 6} = s_8$$

As a result, we can obtain $s_3 \oplus_{Xu} s_5 = s_4 \oplus_{Xu} s_6$ and $s_3 \otimes_{Xu} s_5 = s_4 \otimes_{Xu} s_6$. Comparatively speaking, $s_3 \oplus s_5 < s_4 \oplus s_6$ and $s_3 \otimes s_5 < s_4 \otimes s_6$ may be more easily accepted, and $s_3 \oplus_{Xu} s_5 = s_4 \oplus_{Xu} s_6$ and $s_3 \otimes_{Xu} s_5 = s_4 \otimes_{Xu} s_6$ cannot be accepted.

Definition 3.2. For any $\tilde{s} = \left[s_\alpha, s_\beta\right], \tilde{s}_1 = \left[s_{\alpha_1}, s_{\beta_1}\right], \tilde{s}_2 = \left[s_{\alpha_2}, s_{\beta_2}\right] \in \tilde{S}$, and $\lambda > 0$, their operational laws are defined as follows:

(1) $\tilde{s}_1 \oplus \tilde{s}_2 = \left[s_{\alpha_1}, s_{\beta_1}\right] \oplus \left[s_{\alpha_2}, s_{\beta_2}\right] = \left[s_{\alpha_1} \oplus s_{\alpha_2}, s_{\beta_1} \oplus s_{\beta_2}\right] = \left[s_{f^{-1}\left(f(\alpha_1)+f(\alpha_2)\right)}, s_{f^{-1}\left(f(\beta_1)+f(\beta_2)\right)}\right]$

(2) $\lambda\tilde{s} = \lambda\left[s_\alpha, s_\beta\right] = \left[\lambda s_\alpha, \lambda s_\beta\right] = \left[s_{f^{-1}\left(\lambda f(\alpha)\right)}, s_{f^{-1}\left(\lambda f(\beta)\right)}\right]$

(3) $\tilde{s}_1 \otimes \tilde{s}_2 = \left[s_{\alpha_1}, s_{\beta_1}\right] \otimes \left[s_{\alpha_2}, s_{\beta_2}\right] = \left[s_{\alpha_1} \otimes s_{\alpha_2}, s_{\beta_1} \otimes s_{\beta_2}\right] = \left[s_{g^{-1}\left(g(\alpha_1)+g(\alpha_2)\right)}, s_{g^{-1}\left(g(\beta_1)+g(\beta_2)\right)}\right]$

(4) $\tilde{s}^\lambda = \left[s_\alpha, s_\beta\right]^\lambda = \left[s_\alpha^\lambda, s_\beta^\lambda\right] = \left[s_{g^{-1}\left(\lambda g(\alpha)\right)}, s_{g^{-1}\left(\lambda g(\beta)\right)}\right]$

Especially, if $g(x) = -\log\left(\frac{x}{t}\right)$, then we have:

(5) $\tilde{s}_1 \oplus \tilde{s}_2 = \left[s_{\alpha_1+\alpha_2-\frac{\alpha_1\alpha_2}{t}}, s_{\beta_1+\beta_2-\frac{\beta_1\beta_2}{t}}\right]$ (6) $\lambda\tilde{s} = \left[s_{t\cdot\left[1-\left(1-\frac{\alpha}{t}\right)^\lambda\right]}, s_{t\cdot\left[1-\left(1-\frac{\beta}{t}\right)^\lambda\right]}\right]$

(7) $\tilde{s}_1 \otimes \tilde{s}_2 = \left[s_{\frac{\alpha_1\alpha_2}{t}}, s_{\frac{\beta_1\beta_2}{t}}\right]$ (8) $\tilde{s}^\lambda = \left[s_{t\cdot\left(\frac{\alpha}{t}\right)^\lambda}, s_{t\cdot\left(\frac{\beta}{t}\right)^\lambda}\right]$.

If $g(x) = \log\left(\frac{2t-x}{x}\right)$, then we have:

(9) $\tilde{s}_1 \oplus \tilde{s}_2 = \left[s_{\frac{t^2(\alpha_1+\alpha_2)}{t^2+\alpha_1\alpha_2}}, s_{\frac{t^2(\beta_1+\beta_2)}{t^2+\beta_1\beta_2}}\right]$ (10) $\lambda\tilde{s} = \left[s_{t\cdot\frac{\left(\frac{t+\alpha}{t-\alpha}\right)^\lambda-1}{\left(\frac{t+\alpha}{t-\alpha}\right)^\lambda+1}}, s_{t\cdot\frac{\left(\frac{t+\beta}{t-\beta}\right)^\lambda-1}{\left(\frac{t+\beta}{t-\beta}\right)^\lambda+1}}\right]$

(11) $\tilde{s}_1 \otimes \tilde{s}_2 = \left[s_{\frac{t\alpha_1\alpha_2}{2t^2-t(\alpha_1+\alpha_2)+\alpha_1\alpha_2}}, s_{\frac{t\beta_1\beta_2}{2t^2-t(\beta_1+\beta_2)+\beta_1\beta_2}}\right]$ (12) $\tilde{s}^\lambda = \left[s_{\frac{2t\alpha^\lambda}{\alpha^\lambda+(2t-\alpha)^\lambda}}, s_{\frac{2t\beta^\lambda}{\beta^\lambda+(2t-\beta)^\lambda}}\right]$

If $g(x)=\log\left(\frac{\theta t+(1-\theta)x}{x}\right)$, $\theta>0$, then we have:

(13) $$\tilde{s}_1 \oplus \tilde{s}_2 = \left[s_{\frac{t^2(\alpha_1+\alpha_2)+t(\theta-2)\alpha_1\alpha_2}{t^2+(\theta-1)\alpha_1\alpha_2}}, s_{\frac{t^2(\beta_1+\beta_2)+t(\theta-2)\beta_1\beta_2}{t^2+(\theta-1)\beta_1\beta_2}} \right]$$ (14)

$$\lambda\tilde{s} = \left[s_{\frac{t\left[\left(\frac{t+(\theta-1)\alpha}{t-\alpha}\right)^{\lambda}-1\right]}{\left(\frac{t+(\theta-1)\alpha}{t-\alpha}\right)^{\lambda}+\theta-1}}, s_{\frac{t\left[\left(\frac{t+(\theta-1)\beta}{t-\beta}\right)^{\lambda}-1\right]}{\left(\frac{t+(\theta-1)\beta}{t-\beta}\right)^{\lambda}+\theta-1}} \right];$$

(15) $$\tilde{s}_1 \otimes \tilde{s}_2 = \left[s_{\frac{t\alpha_1\alpha_2}{\theta t^2+(1-\theta)t(\alpha_1+\alpha_2)+\alpha_1\alpha_2(\theta-1)}}, s_{\frac{t\beta_1\beta_2}{\theta t^2+(1-\theta)t(\beta_1+\beta_2)+\beta_1\beta_2(\theta-1)}} \right]$$ (16)

$$\tilde{s}^{\lambda} = \left[s_{\frac{\theta t}{\left(\frac{\theta t+(1-\theta)\alpha}{\alpha}\right)^{\lambda}+\theta-1}}, s_{\frac{\theta t}{\left(\frac{\theta t+(1-\theta)\beta}{\beta}\right)^{\lambda}+\theta-1}} \right]$$

Especially, if $\theta=1$, then (13)–(16) reduce to (5)–(8); if $\theta=2$, then (13)–(16) reduce to (9)–(12).

If $g(x)=\log\left(\frac{\theta-1}{\theta^{\frac{x}{t}}-1}\right)$, $\theta>1$, then we have:

(17) $$\tilde{s}_1 \oplus \tilde{s}_2 = \left[s_{t-t\log_{\theta}\left(1+\frac{\left(\theta^{1-\frac{\alpha_1}{t}}-1\right)\cdot\left(\theta^{1-\frac{\alpha_2}{t}}-1\right)}{\theta-1}\right)}, s_{t-t\log_{\theta}\left(1+\frac{\left(\theta^{1-\frac{\beta_1}{t}}-1\right)\cdot\left(\theta^{1-\frac{\beta_2}{t}}-1\right)}{\theta-1}\right)} \right]$$

(18)

$$\lambda\tilde{s}=\left[s_{t-t\cdot\log_{\theta}\left[\frac{\theta-1+\left(\frac{\theta-1}{\theta^{t-\alpha}-1}\right)^{\lambda}}{\left(\frac{\theta-1}{\theta^{t-\alpha}-1}\right)^{\lambda}}\right]},s_{t-t\cdot\log_{\theta}\left[\frac{\theta-1+\left(\frac{\theta-1}{\theta^{t-\beta}-1}\right)^{\lambda}}{\left(\frac{\theta-1}{\theta^{t-\beta}-1}\right)^{\lambda}}\right]}\right] \quad (19)$$

$$\tilde{s}_1\otimes\tilde{s}_2=\left[s_{t\log_{\theta}\left(1+\frac{\left(\theta^{\frac{\alpha_1}{t}}-1\right)\cdot\left(\theta^{\frac{\alpha_2}{t}}-1\right)}{\theta-1}\right)},s_{t\log_{\theta}\left(1+\frac{\left(\theta^{\frac{\beta_1}{t}}-1\right)\cdot\left(\theta^{\frac{\beta_2}{t}}-1\right)}{\theta-1}\right)}\right]$$

(20) $$\tilde{s}^{\lambda}=\left[s_{t\cdot\log_{\theta}\left[\frac{\theta-1+\left(\frac{\theta-1}{\theta^{\alpha}-1}\right)^{\lambda}}{\left(\frac{\theta-1}{\theta^{\alpha}-1}\right)^{\lambda}}\right]},s_{t\cdot\log_{\theta}\left[\frac{\theta-1+\left(\frac{\theta-1}{\theta^{\beta}-1}\right)^{\lambda}}{\left(\frac{\theta-1}{\theta^{\beta}-1}\right)^{\lambda}}\right]}\right]$$

Especially, if $\theta\to 1$, then (17)–(20) reduce to (5)–(8).

If $g(x)=\frac{1}{x}-\frac{1}{t}$, then we have:

(21) $\tilde{s}_1\oplus\tilde{s}_2=\left[s_{\frac{t^2(\alpha_1+\alpha_2)-2t\alpha_1\alpha_2}{t^2-\alpha_1\alpha_2}},s_{\frac{t^2(\beta_1+\beta_2)-2t\beta_1\beta_2}{t^2-\beta_1\beta_2}}\right]$ (22) $\lambda\tilde{s}=\left[s_{\frac{\lambda t\alpha}{t+(\lambda-1)\alpha}},s_{\frac{\lambda t\beta}{t+(\lambda-1)\beta}}\right]$

(23) $\tilde{s}_1\otimes\tilde{s}_2=\left[s_{\frac{t\alpha_1\alpha_2}{t(\alpha_1+\alpha_2)-\alpha_1\alpha_2}},s_{\frac{t\beta_1\beta_2}{t(\beta_1+\beta_2)-\beta_1\beta_2}}\right]$ (24) $\tilde{s}^{\lambda}=\left[s_{\frac{t\alpha}{\alpha+\lambda(t-\alpha)}},s_{\frac{t\beta}{\beta+\lambda(t-\beta)}}\right]$

If $g(x)=\exp\left(\frac{t-x}{x}\right)-1$, then we have:

(25) $$\tilde{s}_1\oplus\tilde{s}_2=\left[s_{\frac{t\log\left(\exp\left(\frac{\alpha_1}{t-\alpha_1}\right)+\exp\left(\frac{\alpha_2}{t-\alpha_2}\right)-1\right)}{1+\log\left(\exp\left(\frac{\alpha_1}{t-\alpha_1}\right)+\exp\left(\frac{\alpha_2}{t-\alpha_2}\right)-1\right)}},s_{\frac{t\log\left(\exp\left(\frac{\beta_1}{t-\beta_1}\right)+\exp\left(\frac{\beta_2}{t-\beta_2}\right)-1\right)}{1+\log\left(\exp\left(\frac{\beta_1}{t-\beta_1}\right)+\exp\left(\frac{\beta_2}{t-\beta_2}\right)-1\right)}}\right];$$

(26) $$\lambda\tilde{s}=\left[s_{\frac{t\log\left(\lambda\exp\left(\frac{\alpha}{t-\alpha}\right)+1-\lambda\right)}{1+\log\left(\lambda\exp\left(\frac{\alpha}{t-\alpha}\right)+1-\lambda\right)}},s_{\frac{t\log\left(\lambda\exp\left(\frac{\beta}{t-\beta}\right)+1-\lambda\right)}{1+\log\left(\lambda\exp\left(\frac{\beta}{t-\beta}\right)+1-\lambda\right)}}\right];$$

(27) $\tilde{s}_1 \otimes \tilde{s}_2 = \left[s_{\frac{t}{1+\log\left(\exp\left(\frac{t-\alpha_1}{\alpha_1}\right)+\exp\left(\frac{t-\alpha_2}{\alpha_2}\right)-1\right)}}, s_{\frac{t}{1+\log\left(\exp\left(\frac{t-\beta_1}{\beta_1}\right)+\exp\left(\frac{t-\beta_2}{\beta_2}\right)-1\right)}} \right]$;

(28) $\tilde{s}^{\lambda} = \left[s_{\frac{t}{1+\log\left(\lambda\exp\left(\frac{t-\alpha}{\alpha}\right)+1-\lambda\right)}}, s_{\frac{t}{1+\log\left(\lambda\exp\left(\frac{t-\beta}{\beta}\right)+1-\lambda\right)}} \right]$.

Example 3.2. Assume that $t = 8$, $g(x) = \log\left(\frac{2t-x}{x}\right)$, $\tilde{s} = [s_5, s_6]$, $\tilde{s}_1 = [s_1, s_3]$, $\tilde{s}_2 = [s_6, s_7]$, and $\lambda = 2$, then by Definition 3.2. (9)-(12), we have

$$\tilde{s}_1 \oplus \tilde{s}_2 = [s_1, s_3] \oplus [s_6, s_7] = \left[s_{\frac{64\times(1+6)}{64+1\times6}}, s_{\frac{64\times(3+7)}{64+3\times7}} \right] = [s_{6.4000}, s_{7.5294}];$$

$$2\tilde{s} = 2[s_5, s_6] = \left[s_{8\cdot\frac{\left(\frac{8+5}{8-5}\right)^2-1}{\left(\frac{8+5}{8-5}\right)^2+1}}, s_{8\cdot\frac{\left(\frac{8+6}{8-6}\right)^{\lambda}-1}{\left(\frac{8+6}{8-6}\right)^{\lambda}+1}} \right] = [s_{7.1910}, s_{7.6800}];$$

$$\tilde{s}_1 \otimes \tilde{s}_2 = [s_1, s_3] \otimes [s_6, s_7] = \left[s_{\frac{8\times1\times6}{2\times64-8\times(1+6)+1\times6}}, s_{\frac{8\times3\times7}{2\times64-8\times(3+7)+3\times7}} \right] = [s_{0.6154}, s_{2.4348}];$$

$$\tilde{s}^2 = [s_5, s_6]^2 = \left[s_{\frac{2\times8\times5^2}{5^2+(2\times8-5)^2}}, s_{\frac{2\times8\times6^2}{6^2+(2\times8-6)^2}} \right] = [s_{2.7397}, s_{4.2353}].$$

Theorem 3.4. For $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4 \in \tilde{S}$, and $\lambda > 0$, we have the following properties:

(1) $\max\{\tilde{s}_1, \tilde{s}_2\} \le \tilde{s}_1 \oplus \tilde{s}_2 \le [s_t, s_t]$.

$\tilde{s}_1 \oplus \tilde{s}_2 = [s_t, s_t]$ if and only if at least one of $\tilde{s}_1$ and $\tilde{s}_2$ is equal to $[s_t, s_t]$.

$\tilde{s}_1 \oplus \tilde{s}_2 = [s_0, s_0]$ if and only if $\tilde{s}_1 = \tilde{s}_2 = [s_0, s_0]$.

(2) $[s_0, s_0] \le \tilde{s}_1 \otimes \tilde{s}_2 \le \min\{\tilde{s}_1, \tilde{s}_2\}$.

$\tilde{s}_1 \otimes \tilde{s}_2 = [s_0, s_0]$ if and only if at least one of $\tilde{s}_1$ and $\tilde{s}_2$ is equal to $[s_0, s_0]$.

$\tilde{s}_1 \otimes \tilde{s}_2 = [s_t, s_t]$ if and only if $\tilde{s}_1 = \tilde{s}_2 = [s_t, s_t]$.

(3) $[s_0, s_0] \le \lambda\tilde{s}_1 \le [s_t, s_t]$. $\lambda\tilde{s}_1 = [s_0, s_0]$ if and only if $\tilde{s}_1 = [s_0, s_0]$. $\lambda\tilde{s}_1 = [s_t, s_t]$ if and only if $\tilde{s}_1 = [s_t, s_t]$.

(4) $[s_0,s_0] \le \tilde{s}_1^{\lambda} \le [s_t,s_t]$. $[s_0,s_0] = \tilde{s}_1^{\lambda}$ if and only if $\tilde{s}_1 = [s_0,s_0]$. $\tilde{s}_1^{\lambda} = [s_t,s_t]$ if and only if $\tilde{s}_1 = [s_t,s_t]$.
(5) $\tilde{s}_1 \oplus \tilde{s}_2 \le \tilde{s}_3 \oplus \tilde{s}_4$, if $\tilde{s}_1 \le \tilde{s}_3$ and $\tilde{s}_2 \le \tilde{s}_4$; Furthermore, $\tilde{s}_1 \oplus \tilde{s}_2 = \tilde{s}_3 \oplus \tilde{s}_4$ if and only if
(i) $\tilde{s}_1 = \tilde{s}_3$ and $\tilde{s}_2 = \tilde{s}_4$, or
(ii) at least one of $\tilde{s}_1$ and $\tilde{s}_2$ is equal to $[s_t,s_t]$, and at least one of $\tilde{s}_3$ and $\tilde{s}_4$ is equal to $[s_t,s_t]$.
(6) $\tilde{s}_1 \otimes \tilde{s}_2 \le \tilde{s}_3 \otimes \tilde{s}_4$, if $\tilde{s}_1 \le \tilde{s}_3$ and $\tilde{s}_2 \le \tilde{s}_4$; Furthermore, $\tilde{s}_1 \otimes \tilde{s}_2 = \tilde{s}_3 \otimes \tilde{s}_4$ if and only if
(i) $\tilde{s}_1 = \tilde{s}_3$ and $\tilde{s}_2 = \tilde{s}_4$, or
(ii) at least one of $\tilde{s}_1$ and $\tilde{s}_2$ is equal to $[s_0,s_0]$, and at least one of $\tilde{s}_3$ and $\tilde{s}_4$ is equal to $[s_0,s_0]$.

Theorem 3.5. For $\tilde{s}_1,\tilde{s}_2,\tilde{s}_3,\tilde{s}_4 \in \tilde{S}$, and $\lambda,\lambda_1,\lambda_2 \ge 0$, we have the following properties:
(1) $\tilde{s}_1 \oplus \tilde{s}_2 = \tilde{s}_2 \oplus \tilde{s}_1$; (2) $(\tilde{s}_1 \oplus \tilde{s}_2) \oplus \tilde{s}_3 = \tilde{s}_2 \oplus (\tilde{s}_1 \oplus \tilde{s}_3)$; (3) $\tilde{s}_0 \oplus \tilde{s}_1 = [s_0,s_0] \oplus \tilde{s}_1 = \tilde{s}_1$;
(4) $\lambda(\tilde{s}_1 \oplus \tilde{s}_2) = \lambda\tilde{s}_1 \oplus \lambda\tilde{s}_2$; (5) $(\lambda_1+\lambda_2)\tilde{s}_1 = \lambda_1\tilde{s} \oplus \lambda_2\tilde{s}_1$; (6) $\tilde{s}_1 \otimes \tilde{s}_2 = \tilde{s}_2 \otimes \tilde{s}_1$;
(7) $(\tilde{s}_1 \otimes \tilde{s}_2) \otimes \tilde{s}_3 = \tilde{s}_1 \otimes (\tilde{s}_2 \otimes \tilde{s}_3)$; (8) $\tilde{s}_t \otimes \tilde{s}_1 = [s_t,s_t] \otimes \tilde{s}_1 = \tilde{s}_1$; (9) $(\tilde{s}_1 \otimes \tilde{s}_2)^{\lambda} = \tilde{s}_1^{\lambda} \otimes \tilde{s}_2^{\lambda}$;
(10) $\tilde{s}_1^{\lambda_1} \otimes \tilde{s}_1^{\lambda_2} = \tilde{s}_1^{\lambda_1+\lambda_2}$.

Theorem 3.6. Let $\tilde{s},\tilde{s}_1,\tilde{s}_2 \in \tilde{S}$ and $\lambda > 0$, then the following are also valid:
(1) $(\mathrm{neg}(\tilde{s}))^{\lambda} = \mathrm{neg}(\lambda\tilde{s})$. (2) $\lambda\mathrm{neg}(\tilde{s}) = \mathrm{neg}(\tilde{s}^{\lambda})$.
(3) $\mathrm{neg}(\tilde{s}_1) \oplus \mathrm{neg}(\tilde{s}_2) = \mathrm{neg}(\tilde{s}_1 \otimes \tilde{s}_2)$. (4) $\mathrm{neg}(\tilde{s}_1) \otimes \mathrm{neg}(\tilde{s}_2) = \mathrm{neg}(\tilde{s}_1 \oplus \tilde{s}_2)$.

Compared with the Xu's operational laws of uncertain linguistic variables [19], [27] (Definition 2.2), the new operational laws of uncertain linguistic variables have the same advantages as the new operational laws of linguistic terms. An illustrative example is given as follows:

Example 3.3. Assume that there are four uncertain linguistic variables as follows:

$$\tilde{s}_1 = [s_3,s_4],\ \tilde{s}_2 = [s_5,s_6],\ \tilde{s}_3 = [s_4,s_5],\ \tilde{s}_4 = [s_6,s_7].$$

Clearly, $\tilde{s}_1 \prec \tilde{s}_3$ and $\tilde{s}_2 \prec \tilde{s}_4$. Let $g(x) = \log\left(\frac{3-1}{3^{\frac{x}{8}}-1}\right)$, then $f(x) = \log\left(\frac{3-1}{3^{1-\frac{x}{8}}-1}\right)$, $g^{-1}(x) = 8\log_3\left(\frac{3-1+e^x}{e^x}\right)$, $f^{-1}(x) = 8 - 8\log_3\left(\frac{3-1+e^x}{e^x}\right)$. By Definition 3.2 (17) and (19), we have

$$\tilde{s}_1 \oplus \tilde{s}_2 = [s_3,s_4] \oplus [s_5,s_6] = \left[s_{f^{-1}(f(3)+f(5))}, s_{f^{-1}(f(4)+f(6))}\right] = [s_{6.3658}, s_{7.2028}]$$

$$\tilde{s}_1 \otimes \tilde{s}_2 = [s_3,s_4] \otimes [s_5,s_6] = \left[s_{g^{-1}(g(3)+g(5))}, s_{g^{-1}(g(4)+g(6))}\right] = [s_{1.6342}, s_{2.7972}]$$

$$\tilde{s}_3 \oplus \tilde{s}_4 = [s_4, s_5] \oplus [s_6, s_7] = \left[s_{f^{-1}(f(4)+f(6))}, s_{f^{-1}(f(5)+f(7))} \right] = [s_{7.2028}, s_{7.7318}]$$

$$\tilde{s}_3 \otimes \tilde{s}_4 = [s_4, s_5] \otimes [s_6, s_7] = \left[s_{g^{-1}(g(4)+g(6))}, s_{g^{-1}(g(5)+g(7))} \right] = [s_{2.7972}, s_{4.2682}]$$

It is obvious that $\tilde{s}_1 \oplus \tilde{s}_2 \prec \tilde{s}_3 \oplus \tilde{s}_4$ and $\tilde{s}_1 \otimes \tilde{s}_2 \prec \tilde{s}_3 \otimes \tilde{s}_4$, which is consistent with our intuition and can be more easily accepted.

If we use the operations $\oplus_{Xu}$ and $\otimes_{Xu}$ proposed by Xu [6,8] (Definition 2.2) to aggregate $\tilde{s}_1$ and $\tilde{s}_2$, and $\tilde{s}_3$ and $\tilde{s}_4$, respectively, then we have

$$\tilde{s}_1 \oplus_{Xu} \tilde{s}_2 = [s_3, s_4] \oplus_{Xu} [s_5, s_6] = [s_{3+5}, s_{4+6}] = [s_8, s_8]$$

$$\tilde{s}_1 \otimes_{Xu} \tilde{s}_2 = [s_3, s_4] \otimes_{Xu} [s_5, s_6] = [s_{3\times 5}, s_{4\times 6}] = [s_8, s_8]$$

$$\tilde{s}_3 \oplus_{Xu} \tilde{s}_4 = [s_4, s_5] \oplus_{Xu} [s_6, s_7] = [s_{4+6}, s_{5+7}] = [s_8, s_8]$$

$$\tilde{s}_3 \otimes_{Xu} \tilde{s}_4 = [s_4, s_5] \otimes_{Xu} [s_6, s_7] = [s_{4\times 6}, s_{5\times 7}] = [s_8, s_8]$$

As a result, we can obtain

$\tilde{s}_1 \oplus \tilde{s}_2 = \tilde{s}_3 \oplus \tilde{s}_4$ and $\tilde{s}_1 \otimes \tilde{s}_2 = \tilde{s}_3 \otimes \tilde{s}_4$

Comparatively speaking, $\tilde{s}_1 \oplus \tilde{s}_2 \prec \tilde{s}_3 \oplus \tilde{s}_4$ and $\tilde{s}_1 \otimes \tilde{s}_2 \prec \tilde{s}_3 \otimes \tilde{s}_4$ may be more easily accepted, and $\tilde{s}_1 \oplus \tilde{s}_2 = \tilde{s}_3 \oplus \tilde{s}_4$ and $\tilde{s}_1 \otimes \tilde{s}_2 = \tilde{s}_3 \otimes \tilde{s}_4$ cannot be accepted.

3.2 Induced Generalized Uncertain Linguistic Ordered Weighted Averaging (IGULOWA) Operator

The IGOWA operator, however, can only be used in situations where the aggregated arguments are the exact numerical values. In the following, we shall develop an induced generalized uncertain linguistic ordered weighted averaging (IGULOWA) operator to accommodate the situations where the input arguments are given in the form of uncertain linguistic variables.

The IGULOWA operator uses the main characteristics of the ULOWA, IOWA, and GOWA operators. Therefore, it uses uncertain linguistic information represented in the form of uncertain linguistic variables, generalized means, and order-inducing variables. The IGULOWA operator provides a very general formulation that includes as special cases a wide range of aggregation operators, including all the particular cases of the IOWA and GOWA operators, the ULOWA operator, the induced uncertain linguistic ordered weighted averaging (IULOWA) operator, the induced uncertain linguistic ordered weighted harmonic averaging (IULOWHA) operator, the induced uncertain linguistic ordered weighted quadratic averaging (IULOWQA) operator, the generalized uncertain linguistic ordered weighted averaging (GULOWA) operator and the generalized uncertain linguistic weighted averaging (GULWA) operator. The main advantage of the IGULOWA operator is that it considers complex reordering processes that can describe the problem in a more complete way under an uncertain framework that can be assessed with uncertain linguistic variables.

Definition 3.3. An induced generalized uncertain linguistic ordered weighted averaging (IGULOWA) operator $\mathrm{IGULOWA}_\lambda : \tilde{S}^n \to \tilde{S}$ is defined as follows:

$$\mathrm{IGULOWA}_\lambda \left(\langle u_1, \tilde{s}_1 \rangle, \langle u_2, \tilde{s}_2 \rangle, \cdots, \langle u_n, \tilde{s}_n \rangle \right) = \left(\mathop{\oplus}_{j=1}^{n} \left(w_j \tilde{s}_{\sigma(j)}^{\lambda} \right) \right)^{1/\lambda} \tag{7}$$

where $\lambda \in (0,+\infty)$, $w=(w_1,w_2,\cdots,w_n)^T$ is a weighting vector, such that $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$, $\tilde{s}_{\sigma(j)}$ is the $\tilde{s}_i$ value of the ULOWA pair $\langle u_i,\tilde{s}_i\rangle$ having the *j*th largest u_i, and u_i in $\langle u_i,\tilde{s}_i\rangle$ is referred to as the order inducing variable and $\tilde{s}_i$ as the uncertain linguistic argument variable.

However, if there is a tie between $\langle u_i,\tilde{s}_i\rangle$ and $\langle u_j,\tilde{s}_j\rangle$ with respect to order inducing variables such that $u_i = u_j$, in this case, we replace the argument component of each of $\langle u_i,\tilde{s}_i\rangle$ and $\langle u_j,\tilde{s}_j\rangle$ by their generalized mean $\left(\left(\tilde{s}_i^{\lambda} \oplus \tilde{s}_j^{\lambda}\right)/2\right)^{1/\lambda}$ depending on the parameter λ. If k items are tied, we replace these by k replicas of their generalized mean.

Especially, if $\lambda = 1$, then IGULOWA operator reduces to the induced uncertain linguistic ordered weighted averaging (IULOWA) operator, which is shown as follows:

$$\mathrm{IULOWA}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right) = \mathop{\oplus}_{j=1}^{n}\left(w_j\tilde{s}_{\sigma(j)}\right) \tag{8}$$

Theorem 3.7. Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}\right]$ ($i=1,2,\cdots,n$) be a collection of uncertain linguistic variables, then the aggregated value by using the IGULOWA operator is an uncertain linguistic variable, and

$$\mathrm{IGULOWA}_{\lambda}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right) = \left[s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n} w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)}, s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n} w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)}\right] \tag{9}$$

Proof. By using mathematical induction on n, we first prove that

$$\mathop{\oplus}_{j=1}^{n}\left(w_j\tilde{s}_{\sigma(j)}^{\lambda}\right) = \left[s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)}, s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)}\right] \tag{10}$$

For $n=2$, since

$$w_1\tilde{s}_{\sigma(1)}^{\lambda} = \left[s_{f^{-1}\left(w_1 f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(1)}\right)\right)\right)\right)}, s_{f^{-1}\left(w_1 f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(1)}\right)\right)\right)\right)}\right]$$

$$w_2\tilde{s}_{\sigma(2)}^{\lambda} = \left[s_{f^{-1}\left(w_2 f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(2)}\right)\right)\right)\right)}, s_{f^{-1}\left(w_2 f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(2)}\right)\right)\right)\right)}\right]$$

Then we have

$$w_1\tilde{s}^{\lambda}_{\sigma(1)} \oplus w_2\tilde{s}^{\lambda}_{\sigma(2)}$$

$$=\left[s_{f^{-1}\left(w_1 f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(1)}\right)\right)\right)\right)}, s_{f^{-1}\left(w_1 f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(1)}\right)\right)\right)\right)}\right] \oplus \left[s_{f^{-1}\left(w_2 f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(2)}\right)\right)\right)\right)}, s_{f^{-1}\left(w_2 f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(2)}\right)\right)\right)\right)}\right]$$

$$=\left[s_{f^{-1}\left(f\left(f^{-1}\left(w_1 f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(1)}\right)\right)\right)\right)\right)+f\left(f^{-1}\left(w_2 f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(2)}\right)\right)\right)\right)\right)\right)}, s_{f^{-1}\left(f\left(f^{-1}\left(w_1 f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(1)}\right)\right)\right)\right)\right)+f\left(f^{-1}\left(w_2 f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(2)}\right)\right)\right)\right)\right)\right)}\right]$$

$$=\left[s_{f^{-1}\left(w_1 f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(1)}\right)\right)\right)+w_2 f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(2)}\right)\right)\right)\right)}, s_{f^{-1}\left(w_1 f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(1)}\right)\right)\right)+w_2 f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(2)}\right)\right)\right)\right)}\right]$$

That is, the Eq. (10) holds for $n=2$. Suppose that the Eq. (10) holds for $n=k$, i.e.,

$$\mathop{\oplus}_{j=1}^{k}\left(w_j\tilde{s}^{\lambda}_{\sigma(j)}\right)=\left[s_{f^{-1}\left(\sum_{j=1}^{k} w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)}, s_{f^{-1}\left(\sum_{j=1}^{k} w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)}\right]$$

then, when $n=k+1$, we have

$$\mathop{\oplus}_{j=1}^{k+1}\left(w_j\tilde{s}^{\lambda}_{\sigma(j)}\right)=\left(\mathop{\oplus}_{j=1}^{k}\left(w_j\tilde{s}^{\lambda}_{\sigma(j)}\right)\right)\oplus\left(w_{k+1}\tilde{s}^{\lambda}_{\sigma(k+1)}\right)$$

$$=\left[s_{f^{-1}\left(\sum_{j=1}^{k} w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)}, s_{f^{-1}\left(\sum_{j=1}^{k} w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)}\right] \oplus \left[s_{f^{-1}\left(w_{k+1} f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(k+1)}\right)\right)\right)\right)}, s_{f^{-1}\left(w_{k+1} f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(k+1)}\right)\right)\right)\right)}\right]$$

$$=\left[s_{f^{-1}\left(f\left(f^{-1}\left(\sum_{j=1}^{k} w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)+f\left(f^{-1}\left(w_{k+1} f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(k+1)}\right)\right)\right)\right)\right)\right)}, s_{f^{-1}\left(f\left(f^{-1}\left(\sum_{j=1}^{k} w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)+f\left(f^{-1}\left(w_{k+1} f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(k+1)}\right)\right)\right)\right)\right)\right)}\right]$$

$$=\left[s_{f^{-1}\left(\sum_{j=1}^{k} w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)+w_{k+1} f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(k+1)}\right)\right)\right)\right)}, s_{f^{-1}\left(\sum_{j=1}^{k} w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)+w_{k+1} f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(k+1)}\right)\right)\right)\right)}\right]$$

$$=\left[s_{f^{-1}\left(\sum_{j=1}^{k+1} w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)}, s_{f^{-1}\left(\sum_{j=1}^{k+1} w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)}\right]$$

i.e., Eq. (10) holds for $n=k+1$. Thus, Eq. (10) holds for all n.

Furthermore, by Eqs. (7) and (10), we get

$$\mathrm{IGULOWA}_\lambda\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)=\left(\mathop{\oplus}_{j=1}^{n}\left(w_j\tilde{s}_{\sigma(j)}^{\lambda}\right)\right)^{1/\lambda}$$

$$=\left[s_{f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)},s_{f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)}\right]^{\lambda}$$

$$=\left[s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)},s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)}\right]$$

In addition, because $g:[0,t]\to[0,+\infty]$ is a strictly decreasing function and $f(x)=g(t-x)$, $f:[0,t]\to[0,+\infty]$ is a strictly increasing function. Accordingly, $g^{-1}:[0,+\infty]\to[0,t]$ is a strictly decreasing function and $f^{-1}:[0,+\infty]\to[0,t]$ is a strictly increasing function. Moreover, for any $\tilde{s}_i=\left[s_{\alpha_i},s_{\beta_i}\right]\in\tilde{S}$ ($i=1,2,\cdots,n$), we have $s_0\le s_{\alpha_i}\le s_{\beta_i}\le s_t$. Therefore, we have

$$\begin{aligned}s_0&=s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g(0)\right)\right)\right)\right)\right)}\\&\le s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)}\\&\le s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)}\\&\le s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(g^{-1}\left(\lambda g(t)\right)\right)\right)\right)\right)}\\&=s_t\end{aligned}$$

which implies that the aggregated value by using the IGULOWA operator is still an uncertain linguistic variable. This completes the proof of Theorem 3.7.

□

Let $\lambda=1$, then the following result can be easily derived from Theorem 3.7.

Theorem 3.8. Let $\tilde{s}_i=\left[s_{\alpha_i},s_{\beta_i}\right]$ ($i=1,2,\cdots,n$) be a collection of uncertain linguistic variables, then the aggregated value by using the IULOWA operator is an uncertain linguistic variable, and

$$\mathrm{IULOWA}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)=\left[s_{f^{-1}\left(\sum_{j=1}^{n}w_j f\left(\alpha_{\sigma(j)}\right)\right)},s_{f^{-1}\left(\sum_{j=1}^{n}w_j f\left(\beta_{\sigma(j)}\right)\right)}\right] \tag{11}$$

Example 3.4. Let $\langle 0.3,[s_4,s_6]\rangle$, $\langle 0.2,[s_2,s_3]\rangle$, $\langle 0.8,[s_5,s_7]\rangle$, $\langle 0.5,[s_7,s_7]\rangle$ be four ULOWA pairs $\langle u_i,\tilde{s}_i\rangle$. Performing the ordering of the ULOWA pairs with respect to the first component, we have

$$\langle 0.8,[s_5,s_7]\rangle,\ \langle 0.5,[s_7,s_7]\rangle,\ \langle 0.3,[s_4,s_6]\rangle,\ \langle 0.2,[s_2,s_3]\rangle,$$

This ordering induces the ordered uncertain linguistic arguments

$$\tilde{s}_{\sigma(1)}=[s_5,s_7],\ \tilde{s}_{\sigma(2)}=[s_7,s_7],\ \tilde{s}_{\sigma(3)}=[s_4,s_6],\ \tilde{s}_{\sigma(3)}=[s_2,s_3],$$

Suppose that $g(x)=\frac{1}{x}-\frac{1}{8}$, $f(x)=\frac{x}{8(8-x)}$, $g^{-1}(x)=\frac{8}{1+8x}$, $f^{-1}(x)=\frac{64x}{1+8x}$, the weighting vector $w=(0.3,0.4,0.1,0.2)$, and $\lambda=0.7$. Then, by Eq. (9), we have

$$\text{IGULOWA}_{0.7}\left(\langle 0.3,[s_4,s_6]\rangle,\langle 0.2,[s_2,s_3]\rangle,\langle 0.8,[s_5,s_7]\rangle,\langle 0.5,[s_7,s_8]\rangle\right)=[s_{6.2090},s_{6.7342}].$$

Then, we can investigate some desirable properties of the IGULOWA operator.

Theorem 3.9 (Commutativity). If $\left(\langle u_1',\tilde{s}_1'\rangle,\langle u_2',\tilde{s}_2'\rangle,\cdots,\langle u_n',\tilde{s}_n'\rangle\right)$ is any permutation of $\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)$, then we have

$$\text{IGULOWA}_\lambda\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)=\text{IGULOWA}_\lambda\left(\langle u_1',\tilde{s}_1'\rangle,\langle u_2',\tilde{s}_2'\rangle,\cdots,\langle u_n',\tilde{s}_n'\rangle\right) \quad (12)$$

Proof. According to Definition 3.1, let

$$\text{IGULOWA}_\lambda\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)=\left[s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_jf\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)},s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{i=1}^{n}\omega_if\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)}\right]$$

$$\text{IGULOWA}_\lambda\left(\langle u_1',\tilde{s}_1'\rangle,\langle u_2',\tilde{s}_2'\rangle,\cdots,\langle u_n',\tilde{s}_n'\rangle\right)=\left[s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_jf\left(g^{-1}\left(\lambda g\left(\alpha'_{\sigma(j)}\right)\right)\right)\right)\right)\right)},s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{i=1}^{n}\omega_if\left(g^{-1}\left(\lambda g\left(\beta'_{\sigma(j)}\right)\right)\right)\right)\right)\right)}\right]$$

Since $\left(\langle u_1',\tilde{s}_1'\rangle,\langle u_2',\tilde{s}_2'\rangle,\cdots,\langle u_n',\tilde{s}_n'\rangle\right)$ is a permutation of $\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)$, we have $\alpha_{\sigma(j)}=\alpha'_{\sigma(j)}$ and $\beta_{\sigma(j)}=\beta'_{\sigma(j)}$, then

$$\text{IGULOWA}_\lambda\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)=\text{IGULOWA}_\lambda\left(\langle u_1',\tilde{s}_1'\rangle,\langle u_2',\tilde{s}_2'\rangle,\cdots,\langle u_n',\tilde{s}_n'\rangle\right)$$

Theorem 3.10 (Idempotency). If all $\tilde{s}_i=\left[s_{\alpha_i},s_{\beta_i}\right]$ ($i=1,2,\cdots,n$) are equal, i.e., $\tilde{s}_i=\tilde{s}=\left[s_\alpha,s_\beta\right]$, for all i, then

$$\text{IGULOWA}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)=\tilde{s} \quad (13)$$

Proof. Let $\tilde{s}_i=\tilde{s}=\left[s_\alpha,s_\beta\right]$, then we have

$$\mathrm{IGULOWA}_{\lambda}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)=\mathrm{IGULOWA}_{\lambda}\left(\langle u_1,\tilde{s}\rangle,\langle u_2,\tilde{s}\rangle,\cdots,\langle u_n,\tilde{s}\rangle\right)$$

$$=\left[s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}(\lambda g(\alpha))\right)\right)\right)\right)},s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(g^{-1}(\lambda g(\beta))\right)\right)\right)\right)}\right]$$

$$=\left[s_{\alpha},s_{\beta}\right]$$

This completes the proof of Theorem 3.10. □

Theorem 3.11 (Monotonicity). Let $\tilde{s}_i=\left[s_{\alpha_i},s_{\beta_i}\right]$ and $\tilde{s}'_i=\left[s_{\alpha'_i},s_{\beta'_i}\right]$ ($i=1,2,\cdots,n$) be two collections of uncertain linguistic variables, if $s_{\alpha_i}\le s_{\alpha'_i}$ and $s_{\beta_i}\le s_{\beta'_i}$, for all i, then

$$\mathrm{IGULOWA}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)\le\mathrm{IGULOWA}\left(\langle u_1,\tilde{s}'_1\rangle,\langle u_2,\tilde{s}'_2\rangle,\cdots,\langle u_n,\tilde{s}'_n\rangle\right)\quad(14)$$

Proof. If $s_{\alpha_i}\le s_{\alpha'_i}$ and $s_{\beta_i}\le s_{\beta'_i}$, for all i, then $s_{\alpha_{\sigma(i)}}\le s_{\alpha'_{\sigma(i)}}$ and $s_{\beta_{\sigma(i)}}\le s_{\beta'_{\sigma(i)}}$. Therefore,

$$g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)\le g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\alpha'_{\sigma(j)}\right)\right)\right)\right)\right)\right)$$

and

$$g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)\le g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\beta'_{\sigma(j)}\right)\right)\right)\right)\right)\right)$$

By Definition 2.3, we have

$$s\left(\mathrm{IGULOWA}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)\right)$$

$$=\frac{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)+g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)}{2}$$

$$\le\frac{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\alpha'_{\sigma(j)}\right)\right)\right)\right)\right)\right)+g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\beta'_{\sigma(j)}\right)\right)\right)\right)\right)\right)}{2}$$

$$\le s\left(\mathrm{IGULOWA}\left(\langle u_1,\tilde{s}'_1\rangle,\langle u_2,\tilde{s}'_2\rangle,\cdots,\langle u_n,\tilde{s}'_n\rangle\right)\right)$$

If $s\left(\mathrm{IGULOWA}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)\right)<s\left(\mathrm{IGULOWA}\left(\langle u_1,\tilde{s}'_1\rangle,\langle u_2,\tilde{s}'_2\rangle,\cdots,\langle u_n,\tilde{s}'_n\rangle\right)\right)$, then by Definition 2.5, we have $\mathrm{IGULOWA}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)<\mathrm{IGULOWA}\left(\langle u_1,\tilde{s}'_1\rangle,\langle u_2,\tilde{s}'_2\rangle,\cdots,\langle u_n,\tilde{s}'_n\rangle\right)$.

If $s\left(\text{IGULOWA}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)\right)=s\left(\text{IGULOWA}\left(\langle u_1,\tilde{s}_1'\rangle,\langle u_2,\tilde{s}_2'\rangle,\cdots,\langle u_n,\tilde{s}_n'\rangle\right)\right)$, then we have

$$\frac{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)+g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)}{2}$$

$$=\frac{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\alpha'_{\sigma(j)}\right)\right)\right)\right)\right)\right)+g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\beta'_{\sigma(j)}\right)\right)\right)\right)\right)\right)}{2}$$

which implies that

$$g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)=g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\alpha'_{\sigma(j)}\right)\right)\right)\right)\right)\right)$$

and

$$g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)=g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\beta'_{\sigma(j)}\right)\right)\right)\right)\right)\right)$$

Thus, by Definition 2.4,

$$v\left(\text{IGULOWA}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)\right)$$

$$=\frac{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)-g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)}{2}$$

$$=\frac{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\beta'_{\sigma(j)}\right)\right)\right)\right)\right)\right)-g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\alpha'_{\sigma(j)}\right)\right)\right)\right)\right)\right)}{2}$$

$$=v\left(\text{IGULOWA}\left(\langle u_1,\tilde{s}_1'\rangle,\langle u_2,\tilde{s}_2'\rangle,\cdots,\langle u_n,\tilde{s}_n'\rangle\right)\right)$$

Based on the above analysis, according to Definition 2.5, we can obtain

$$\text{IGULOWA}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)\le\text{IGULOWA}\left(\langle u_1,\tilde{s}_1'\rangle,\langle u_2,\tilde{s}_2'\rangle,\cdots,\langle u_n,\tilde{s}_n'\rangle\right).$$

This completes the proof of Theorem 3.11. □

Theorem 3.12 (Boundedness). Let $\tilde{s}_i=\left[s_{\alpha_i},s_{\beta_i}\right]$ ($i=1,2,\cdots,n$) be a collection of uncertain linguistic variables, then

$$\min_i\{\tilde{s}_i\}\le\text{IGULOWA}_\lambda\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)\le\max_i\{\tilde{s}_i\}\tag{15}$$

Proof. Because $\min_{1\le i\le n}\left\{s_{\alpha_i}\right\}\le s_{\alpha_i}\le\max_{1\le i\le n}\left\{s_{\alpha_i}\right\}$ and $\min_{1\le i\le n}\left\{s_{\beta_i}\right\}\le s_{\beta_i}\le\max_{1\le i\le n}\left\{s_{\beta_i}\right\}$,

$$\begin{aligned}\min_{1\le j\le n}\{\alpha_j\} &= g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\min_{1\le j\le n}\{\alpha_j\}\right)\right)\right)\right)\right)\right)\\ &\le g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)\\ &\le g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\max_{1\le j\le n}\{\alpha_j\}\right)\right)\right)\right)\right)\right)\\ &= \max_{1\le j\le n}\{\alpha_j\}\end{aligned}$$

$$\begin{aligned}\min_{1\le j\le n}\{\beta_j\} &= g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\min_{1\le j\le n}\{\beta_j\}\right)\right)\right)\right)\right)\right)\\ &\le g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right) \quad)\\ &\le g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\max_{1\le j\le n}\{\beta_j\}\right)\right)\right)\right)\right)\right)\\ &= \max_{1\le j\le n}\{\beta_j\}\end{aligned}$$

Let $\mathrm{IGULOWA}_\lambda\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)=\tilde{s}=\left[s_\alpha,s_\beta\right]$. Then,

$$\frac{\min_{1\le j\le n}\{\alpha_j\}+\min_{1\le j\le n}\{\beta_j\}}{2t}\le s(\tilde{s})=\frac{\alpha+\beta}{2t}\le\frac{\max_{1\le j\le n}\{\alpha_j\}+\max_{1\le j\le n}\{\beta_j\}}{2t}$$

If $\frac{\min_{1\le j\le n}\{\alpha_j\}+\min_{1\le j\le n}\{\beta_j\}}{2t}< s(\tilde{s})=\frac{\alpha+\beta}{2t}<\frac{\max_{1\le j\le n}\{\alpha_j\}+\max_{1\le j\le n}\{\beta_j\}}{2t}$, then by Definition 2.5,

$$\min_i\{\tilde{s}_i\}\le\mathrm{IGULOWA}_\lambda\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)\le\max_i\{\tilde{s}_i\}.$$

If $s(\tilde{s})=\frac{\alpha+\beta}{2t}=\frac{\max_{1\le j\le n}\{\alpha_j\}+\max_{1\le j\le n}\{\beta_j\}}{2t}$, then $\alpha=\max_{1\le j\le n}\{\alpha_j\}$, $\beta=\max_{1\le j\le n}\{\beta_j\}$, thus, $v(\tilde{s})=\frac{\beta-\alpha}{2t}=\frac{\max_{1\le j\le n}\{\beta_j\}-\max_{1\le j\le n}\{\alpha_j\}}{2t}$, which implies that $\mathrm{IGULOWA}_\lambda\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)=\max_i\{\tilde{s}_i\}$.

If $s(\tilde{s})=\frac{\alpha+\beta}{2t}=\frac{\min\limits_{1\le j\le n}\{\alpha_j\}+\min\limits_{1\le j\le n}\{\beta_j\}}{2t}$, then $\alpha=\min\limits_{1\le j\le n}\{\alpha_j\}$, $\beta=\min\limits_{1\le j\le n}\{\beta_j\}$, thus, $v(\tilde{s})=\frac{\beta-\alpha}{2t}=\frac{\min\limits_{1\le j\le n}\{\beta_j\}-\min\limits_{1\le j\le n}\{\alpha_j\}}{2t}$, which implies that $\text{IGULOWA}_\lambda\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)=\min\limits_i\{\tilde{s}_i\}$.

From the above analysis, we can conclude that Eq. (15) always holds.

Theorem 3.13. Let $\tilde{s}_i=\left[s_{\alpha_i},s_{\beta_i}\right]$ ($i=1,2,\cdots,n$) be a collection of uncertain linguistic variables, and $\tilde{s}=\left[s_\alpha,s_\beta\right]$ is an uncertain linguistic variable, then

$$\text{IULOWA}\left(\langle u_1,\tilde{s}_1\oplus\tilde{s}\rangle,\langle u_2,\tilde{s}_2\oplus\tilde{s}\rangle,\cdots,\langle u_n,\tilde{s}_n\oplus\tilde{s}\rangle\right)=\text{IULOWA}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)\oplus\tilde{s}$$
(16)

Proof. Since

$$\tilde{s}_i\oplus\tilde{s}=\left[s_{f^{-1}(f(\alpha_i)+f(\alpha))},s_{f^{-1}(f(\beta_i)+f(\beta))}\right]$$

we have

$$\begin{aligned}\text{IULOWA}\left(\langle u_1,\tilde{s}_1\oplus\tilde{s}\rangle,\langle u_2,\tilde{s}_2\oplus\tilde{s}\rangle,\cdots,\langle u_n,\tilde{s}_n\oplus\tilde{s}\rangle\right)&=\left[s_{f^{-1}\left(\sum_{j=1}^n w_j f\left(f^{-1}\left(f\left(\alpha_{\sigma(j)}\right)+f(\alpha)\right)\right)\right)},s_{f^{-1}\left(\sum_{j=1}^n w_j f\left(f^{-1}\left(f\left(\beta_{\sigma(j)}\right)+f(\beta)\right)\right)\right)}\right]\\&=\left[s_{f^{-1}\left(\sum_{j=1}^n w_j\left(f\left(\alpha_{\sigma(j)}\right)+f(\alpha)\right)\right)},s_{f^{-1}\left(\sum_{j=1}^n w_j\left(f\left(\beta_{\sigma(j)}\right)+f(\beta)\right)\right)}\right]\\&=\left[s_{f^{-1}\left(f(\alpha)+\sum_{j=1}^n w_j f\left(\alpha_{\sigma(j)}\right)\right)},s_{f^{-1}\left(f(\beta)+\sum_{j=1}^n w_j f\left(\beta_{\sigma(j)}\right)\right)}\right]\end{aligned}$$

and

$$\begin{aligned}\text{IULOWA}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)\oplus\tilde{s}&=\left[s_{f^{-1}\left(\sum_{j=1}^n w_j f\left(\alpha_{\sigma(j)}\right)\right)},s_{f^{-1}\left(\sum_{j=1}^n w_j f\left(\beta_{\sigma(j)}\right)\right)}\right]\oplus\left[s_\alpha,s_\beta\right]\\&=\left[s_{f^{-1}\left(f\left(f^{-1}\left(\sum_{j=1}^n w_j f\left(\alpha_{\sigma(j)}\right)\right)\right)+f(\alpha)\right)},s_{f^{-1}\left(f\left(f^{-1}\left(\sum_{j=1}^n w_j f\left(\beta_{\sigma(j)}\right)\right)\right)+f(\beta)\right)}\right]\\&=\left[s_{f^{-1}\left(f(\alpha)+\sum_{j=1}^n w_j f\left(\alpha_{\sigma(j)}\right)\right)},s_{f^{-1}\left(f(\beta)+\sum_{j=1}^n w_j f\left(\beta_{\sigma(j)}\right)\right)}\right]\end{aligned}$$

which completes the proof.□□

Theorem 3.14. Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}\right]$ ($i = 1, 2, \cdots, n$) be a collection of uncertain linguistic variables, if $r > 0$, then

$$\mathrm{IULOWA}\left(\langle u_1, r\tilde{s}_1\rangle, \langle u_2, r\tilde{s}_2\rangle, \cdots, \langle u_n, r\tilde{s}_n\rangle\right) = r\mathrm{IULOWA}\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right)$$
(17)

Proof. Since for any $i = 1, 2, \cdots, n$,

$$r\tilde{s}_i = \left[s_{f^{-1}(rf(\alpha_i))}, s_{f^{-1}(rf(\beta_i))}\right]$$

Based on Definition 3.2 and Eq. (11), we have

$$\mathrm{IULOWA}\left(\langle u_1, r\tilde{s}_1\rangle, \langle u_2, r\tilde{s}_2\rangle, \cdots, \langle u_n, r\tilde{s}_n\rangle\right) = \left[s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(f^{-1}\left(rf\left(\alpha_{\sigma(j)}\right)\right)\right)\right)}, s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(f^{-1}\left(rf\left(\beta_{\sigma(j)}\right)\right)\right)\right)}\right]$$

$$= \left[s_{f^{-1}\left(r\sum_{j=1}^{n} w_j f\left(\alpha_{\sigma(j)}\right)\right)}, s_{f^{-1}\left(r\sum_{j=1}^{n} w_j f\left(\beta_{\sigma(j)}\right)\right)}\right]$$

and

$$r\mathrm{IULOWA}\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) = r\left[s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\alpha_{\sigma(j)}\right)\right)}, s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\beta_{\sigma(j)}\right)\right)}\right]$$

$$= \left[s_{f^{-1}\left(rf\left(f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\alpha_{\sigma(j)}\right)\right)\right)\right)}, s_{f^{-1}\left(rf\left(f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\beta_{\sigma(j)}\right)\right)\right)\right)}\right]$$

$$= \left[s_{f^{-1}\left(r\sum_{j=1}^{n} w_j f\left(\alpha_{\sigma(j)}\right)\right)}, s_{f^{-1}\left(r\sum_{j=1}^{n} w_j f\left(\beta_{\sigma(j)}\right)\right)}\right]$$

This completes the proof of Theorem 3.14. □

According to Theorems 3.13 and 3.14, we can easily get the following result.

Theorem 3.15. Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}\right]$ ($i = 1, 2, \cdots, n$) be a collection of uncertain linguistic variables, if $r > 0$ and $\tilde{s} = \left[s_\alpha, s_\beta\right]$ is an uncertain linguistic variable, then

$$\mathrm{IULOWA}\left(\langle u_1, r\tilde{s}_1 \oplus \tilde{s}\rangle, \langle u_2, r\tilde{s}_2 \oplus \tilde{s}\rangle, \cdots, \langle u_n, r\tilde{s}_n \oplus \tilde{s}\rangle\right) = r\mathrm{IULOWA}\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) \oplus \tilde{s}$$ (18)

Theorem 3.16. Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}\right]$ and $\tilde{s}_i' = \left[s_{\alpha_i'}, s_{\beta_i'}\right]$ ($i = 1, 2, \cdots, n$) be two collections of uncertain linguistic variables, then

$$\begin{aligned} &\mathrm{IULOWA}\left(\langle u_1, \tilde{s}_1 \oplus \tilde{s}_1'\rangle, \langle u_2, \tilde{s}_2 \oplus \tilde{s}_2'\rangle, \cdots, \langle u_n, \tilde{s}_n \oplus \tilde{s}_n'\rangle\right) \\ &= \mathrm{IULOWA}\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) \oplus \mathrm{IULOWA}\left(\langle u_1, \tilde{s}_1'\rangle, \langle u_2, \tilde{s}_2'\rangle, \cdots, \langle u_n, \tilde{s}_n'\rangle\right) \end{aligned}$$ (19)

Proof. According to Definition 3.1, we have

$$\tilde{s}_i \oplus \tilde{s}_i' = \left[s_{f^{-1}\left(f(\alpha_i)+f(\alpha_i')\right)}, s_{f^{-1}\left(f(\beta_i)+f(\beta_i')\right)} \right]$$

then

$$\mathrm{IULOWA}\left(\left\langle u_1, \tilde{s}_1 \oplus \tilde{s}_1' \right\rangle, \left\langle u_2, \tilde{s}_2 \oplus \tilde{s}_2' \right\rangle, \cdots, \left\langle u_n, \tilde{s}_n \oplus \tilde{s}_n' \right\rangle\right)$$

$$= \left[s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(f^{-1}\left(f\left(\alpha_{\sigma(j)}\right)+f\left(\alpha_{\sigma(j)}'\right)\right)\right)\right)}, s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(f^{-1}\left(f\left(\beta_{\sigma(j)}\right)+f\left(\beta_{\sigma(j)}'\right)\right)\right)\right)} \right]$$

$$= \left[s_{f^{-1}\left(\sum_{j=1}^{n} w_j \left(f\left(\alpha_{\sigma(j)}\right)+f\left(\alpha_{\sigma(j)}'\right)\right)\right)}, s_{f^{-1}\left(\sum_{j=1}^{n} w_j \left(f\left(\beta_{\sigma(j)}\right)+f\left(\beta_{\sigma(j)}'\right)\right)\right)} \right]$$

$$= \left[s_{f^{-1}\left(\left(\sum_{j=1}^{n} w_j f\left(\alpha_{\sigma(j)}\right)\right)+\left(\sum_{j=1}^{n} w_j f\left(\alpha_{\sigma(j)}'\right)\right)\right)}, s_{f^{-1}\left(\left(\sum_{j=1}^{n} w_j f\left(\beta_{\sigma(j)}\right)\right)+\left(\sum_{j=1}^{n} w_j f\left(\beta_{\sigma(j)}'\right)\right)\right)} \right]$$

and

$$\mathrm{IULOWA}\left(\left\langle u_1, \tilde{s}_1 \right\rangle, \left\langle u_2, \tilde{s}_2 \right\rangle, \cdots, \left\langle u_n, \tilde{s}_n \right\rangle\right) \oplus \mathrm{IULOWA}\left(\left\langle u_1, \tilde{s}_1' \right\rangle, \left\langle u_2, \tilde{s}_2' \right\rangle, \cdots, \left\langle u_n, \tilde{s}_n' \right\rangle\right)$$

$$= \left[s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\alpha_{\sigma(j)}\right)\right)}, s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\beta_{\sigma(j)}\right)\right)} \right] \oplus \left[s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\alpha_{\sigma(j)}'\right)\right)}, s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\beta_{\sigma(j)}'\right)\right)} \right]$$

$$= \left[s_{f^{-1}\left(f\left(f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\alpha_{\sigma(j)}\right)\right)\right)+f\left(f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\alpha_{\sigma(j)}'\right)\right)\right)\right)}, s_{f^{-1}\left(f\left(f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\beta_{\sigma(j)}\right)\right)\right)+f\left(f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\beta_{\sigma(j)}'\right)\right)\right)\right)} \right]$$

$$= \left[s_{f^{-1}\left(\left(\sum_{j=1}^{n} w_j f\left(\alpha_{\sigma(j)}\right)\right)+\left(\sum_{j=1}^{n} w_j f\left(\alpha_{\sigma(j)}'\right)\right)\right)}, s_{f^{-1}\left(\left(\sum_{j=1}^{n} w_j f\left(\beta_{\sigma(j)}\right)\right)+\left(\sum_{j=1}^{n} w_j f\left(\beta_{\sigma(j)}'\right)\right)\right)} \right]$$

which completes the proof. □

We now look at some special cases of the IGULOWA operator obtained by using different choices of the parameters: the associated weighting vector w, the order inducing variable u_j, and the parameter λ.

(1) If $\lambda = 1$, then the IGULOWA operator reduces to the following formula:

$$\mathrm{IULOWA}\left(\left\langle u_1, \tilde{s}_1 \right\rangle, \left\langle u_2, \tilde{s}_2 \right\rangle, \cdots, \left\langle u_n, \tilde{s}_n \right\rangle\right) = \left[s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\alpha_{\sigma(j)}\right)\right)}, s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\beta_{\sigma(j)}\right)\right)} \right] \quad (20)$$

which is called an induced uncertain linguistic ordered weighted averaging (IULOWA) operator, where $\tilde{s}_{\sigma(j)}$ is presented as in Definition 3.3.

(2) If $\lambda = -1$, then the IGULOWA operator reduces to the following formula:

$$\mathrm{IULOWHA}\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) = \left[s_{g^{-1}\left(-g\left(f^{-1}\left(\sum_{j=1}^{n} w_j f\left(g^{-1}\left(-g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)}, s_{g^{-1}\left(-g\left(f^{-1}\left(\sum_{j=1}^{n} w_j f\left(g^{-1}\left(-g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)} \right] \quad (21)$$

which is called an induced uncertain linguistic ordered weighted harmonic averaging (IULOWHA) operator.

(3) If $\lambda = 2$, then the IGULOWA operator reduces to the following formula:

$$\mathrm{IULOWQA}\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) = \left[s_{g^{-1}\left(2g\left(f^{-1}\left(\sum_{j=1}^{n} w_j f\left(g^{-1}\left(2g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)}, s_{g^{-1}\left(2g\left(f^{-1}\left(\sum_{j=1}^{n} w_j f\left(g^{-1}\left(2g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)} \right] \quad (22)$$

which is called an induced uncertain linguistic ordered weighted quadratic averaging (IULOWQA) operator.

(4) If $u_j = \tilde{s}_j$ for all j, then the IGULOWA operator reduces to the following formula:

$$\mathrm{GULOWA}_\lambda\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) = \left[s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n} w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)}, s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n} w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)} \right] \quad (23)$$

which is called an generalized uncertain linguistic ordered weighted averaging (GULOWA) operator, where $\tilde{s}_{\sigma(j)}$ is the *j*th largest of the $\tilde{s}_i$.

(5) If $u_j = \tilde{s}_j$ for all j and $\lambda = 1$, then the IGULOWA operator reduces to the following formula:

$$\mathrm{ULOWA}\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) = \mathop{\oplus}_{j=1}^{n}\left(w_j \tilde{s}_{\sigma(j)}\right) = \left[s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\alpha_{\sigma(j)}\right)\right)}, s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\beta_{\sigma(j)}\right)\right)} \right] \quad (24)$$

which is called an uncertain linguistic ordered weighted averaging (ULOWA) operator, where $\tilde{s}_{\sigma(j)}$ is the *j*th largest of the $\tilde{s}_i$.

(6) If $u_j = -j$ for all j and the associated weighting vector $w = \left(w_1, w_2, \cdots, w_n\right)^T$ is the weight vector of $\tilde{s}_j$ ($j = 1, 2, \cdots, n$), then the IGULOWA operator reduces to the following formula:

$$\mathrm{GULWA}_\lambda\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) = \left[s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n} w_j f\left(g^{-1}\left(\lambda g\left(\alpha_j\right)\right)\right)\right)\right)\right)}, s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n} w_j f\left(g^{-1}\left(\lambda g\left(\beta_j\right)\right)\right)\right)\right)\right)}\right] \quad (25)$$

which is called an generalized uncertain linguistic weighted averaging (GULWA) operator.

(7) If $u_j = -j$ for all j, the associated weighting vector $w = \left(w_1, w_2, \cdots, w_n\right)^T$ is the weight vector of $\tilde{s}_j$ ($j = 1, 2, \cdots, n$), and $\lambda = 1$, then the IGULOWA operator reduces to the following formula:

$$\mathrm{ULWA}_\lambda\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) = \bigoplus_{j=1}^{n}\left(w_j \tilde{s}_j\right) = \left[s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\alpha_j\right)\right)}, s_{f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\beta_j\right)\right)}\right] \quad (26)$$

which is called an uncertain linguistic weighted averaging (ULWA) operator.

(8) If $w = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^T$, then the IGULOWA operator reduces to the following formula:

$$\mathrm{IGULOA}_\lambda\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) = \left[s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n} \frac{1}{n} f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)}, s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n} \frac{1}{n} f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)}\right] \quad (27)$$

which is called an induced generalized uncertain linguistic ordered averaging (IGULOA) operator.

(9) If $w = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^T$ and $\lambda = 1$, then the IGULOWA operator reduces to the following formula:

$$\mathrm{IULOA}\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) = \left[s_{f^{-1}\left(\sum_{j=1}^{n} \frac{1}{n} f\left(\alpha_{\sigma(j)}\right)\right)}, s_{f^{-1}\left(\sum_{j=1}^{n} \frac{1}{n} f\left(\beta_{\sigma(j)}\right)\right)}\right] \quad (28)$$

which is called an induced uncertain linguistic ordered averaging (IULOA) operator.

3.3. Induced Generalized Uncertain Linguistic Ordered Weighted Geometric (IGULOWG) Operator

Based on the IGULOWA operator and the geometric mean, here we define an induced generalized uncertain linguistic ordered weighted geometric (IGULOWG) operator:

Definition 3.4. An induced generalized uncertain linguistic ordered weighted geometric (IGULOWG) operator $\mathrm{IGULOWG}_{\lambda}: \tilde{S}^n \to \tilde{S}$ is defined as follows:

$$\mathrm{IGULOWG}_{\lambda}\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) = \frac{1}{\lambda}\left(\bigotimes_{j=1}^{n}\left(\lambda \tilde{s}_{\sigma(j)}\right)^{w_j}\right) \tag{29}$$

where $\lambda \in (0, +\infty)$, $w = (w_1, w_2, \cdots, w_n)^T$ is a weighting vector, such that $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$, $\tilde{s}_{\sigma(j)}$ is the $\tilde{s}_i$ value of the ULOWG pair $\langle u_i, \tilde{s}_i\rangle$ having the *j*th largest u_i, and u_i in $\langle u_i, \tilde{s}_i\rangle$ is referred to as the order inducing variable and $\tilde{s}_i$ as the uncertain linguistic argument variable.

However, if there is a tie between $\langle u_i, \tilde{s}_i\rangle$ and $\langle u_j, \tilde{s}_j\rangle$ with respect to order inducing variables such that $u_i = u_j$, in this case, we replace the argument component of each of $\langle u_i, \tilde{s}_i\rangle$ and $\langle u_j, \tilde{s}_j\rangle$ by their generalized geometric mean $\frac{1}{\lambda}\left((\lambda \tilde{s}_i)^{\frac{1}{2}} \otimes (\lambda \tilde{s}_j)^{\frac{1}{2}}\right)$ depending on the parameter λ. If k items are tied, we replace these by k replicas of their generalized geometric mean.

Especially, if $\lambda = 1$, then IGULOWG operator reduces to the induced uncertain linguistic ordered weighted geometric (IULOWG) operator, which is shown as follows:

$$\mathrm{IULOWG}\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) = \bigotimes_{j=1}^{n} \tilde{s}_{\sigma(j)}^{w_j} \tag{30}$$

Theorem 3.17. Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}\right]$ ($i = 1, 2, \cdots, n$) be a collection of uncertain linguistic variables, then the aggregated value by using the IGULOWG operator is an uncertain linguistic variable, and

$$\mathrm{IGULOWG}_{\lambda}\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) = \left[s_{f^{-1}\left(\frac{1}{\lambda} f\left(g^{-1}\left(\sum_{j=1}^{n} w_j g\left(f^{-1}\left(\lambda f\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)}, s_{f^{-1}\left(\frac{1}{\lambda} f\left(g^{-1}\left(\sum_{j=1}^{n} w_j g\left(f^{-1}\left(\lambda f\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)} \right] \tag{31}$$

Let $\lambda = 1$, then the following result can be easily derived from Theorem 3.17.

Theorem 3.18. Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}\right]$ ($i = 1, 2, \cdots, n$) be a collection of uncertain linguistic variables, then the aggregated value by using the IULOWG operator is an uncertain linguistic variable, and

$$\text{IULOWG}\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) = \left[s_{g^{-1}\left(\sum_{j=1}^{n} w_j g\left(\alpha_{\sigma(j)}\right)\right)}, s_{g^{-1}\left(\sum_{j=1}^{n} w_j g\left(\beta_{\sigma(j)}\right)\right)}\right] \quad (32)$$

Example 3.5. Suppose that we have four ULOWG pairs $\langle u_i, \tilde{s}_i\rangle$ given $\langle s_2, [s_5, s_6]\rangle$, $\langle s_5, [s_4, s_5]\rangle$, $\langle s_3, [s_1, s_3]\rangle$, $\langle s_6, [s_5, s_7]\rangle$. Performing the ordering of the ULOWG pairs with respect to the first component, we have

$$\langle s_6, [s_5, s_7]\rangle,\ \langle s_5, [s_4, s_5]\rangle,\ \langle s_3, [s_1, s_3]\rangle,\ \langle s_2, [s_5, s_6]\rangle$$

This ordering induces the ordered uncertain linguistic arguments

$$[s_5, s_7],\ [s_4, s_5],\ [s_1, s_3],\ [s_5, s_6]$$

If the weighting vector $w = (0.3, 0.1, 0.5, 0.1)$, $g(x) = \log\left(\dfrac{2t - x}{x}\right)$, $f(x) = \log\left(\dfrac{t + x}{t - x}\right)$, $g^{-1}(x) = \dfrac{2t}{1 + e^x}$, $f^{-1}(x) = \dfrac{(e^x - 1)t}{e^x + 1}$, and $\lambda = 5$, then by Eq. (31), we get

$$\text{IGULOWG}_5\left(\langle s_2, [s_5, s_6]\rangle, \langle s_5, [s_4, s_5]\rangle, \langle s_3, [s_1, s_3]\rangle, \langle s_6, [s_5, s_7]\rangle\right) = [s_{1.5918}, s_{3.4649}].$$

Then we can investigate some desirable properties of the IGULOWG operator.

Theorem 3.19 (Commutativity). If $\left(\langle u_1', \tilde{s}_1'\rangle, \langle u_2', \tilde{s}_2'\rangle, \cdots, \langle u_n', \tilde{s}_n'\rangle\right)$ is any permutation of $\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right)$, then we have

$$\text{IGULOWG}_\lambda\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) = \text{IGULOWG}_\lambda\left(\langle u_1', \tilde{s}_1'\rangle, \langle u_2', \tilde{s}_2'\rangle, \cdots, \langle u_n', \tilde{s}_n'\rangle\right) \quad (33)$$

Theorem 3.20 (Idempotency). If all $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}\right]$ ($i = 1, 2, \cdots, n$) are equal, i.e., $\tilde{s}_i = \tilde{s} = \left[s_\alpha, s_\beta\right]$, for all i, then

$$\text{IGULOWG}\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) = \tilde{s} \quad (34)$$

Theorem 3.21 (Monotonicity). Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}\right]$ and $\tilde{s}_i' = \left[s_{\alpha_i'}, s_{\beta_i'}\right]$ ($i = 1, 2, \cdots, n$) be two collections of uncertain linguistic variables, if $s_{\alpha_i} \le s_{\alpha_i'}$ and $s_{\beta_i} \le s_{\beta_i'}$, for all i, then

$$\text{IGULOWG}\left(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \cdots, \langle u_n, \tilde{s}_n\rangle\right) \le \text{IGULOWG}\left(\langle u_1, \tilde{s}_1'\rangle, \langle u_2, \tilde{s}_2'\rangle, \cdots, \langle u_n, \tilde{s}_n'\rangle\right) \quad (35)$$

Theorem 3.22 (Boundedness). Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i} \right]$ ($i = 1, 2, \cdots, n$) be a collection of uncertain linguistic variables, then

$$\min_i \{\tilde{s}_i\} \le \mathrm{IGULOWG}_\lambda \left(\langle u_1, \tilde{s}_1 \rangle, \langle u_2, \tilde{s}_2 \rangle, \cdots, \langle u_n, \tilde{s}_n \rangle \right) \le \max_i \{\tilde{s}_i\} \tag{36}$$

Theorem 3.23. Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i} \right]$ ($i = 1, 2, \cdots, n$) be a collection of uncertain linguistic variables, and $\tilde{s} = \left[s_\alpha, s_\beta \right]$ is an uncertain linguistic variable, then

$$\mathrm{IULOWG}\left(\langle u_1, \tilde{s}_1 \otimes \tilde{s} \rangle, \langle u_2, \tilde{s}_2 \otimes \tilde{s} \rangle, \cdots, \langle u_n, \tilde{s}_n \otimes \tilde{s} \rangle \right) = \mathrm{IULOWG}\left(\langle u_1, \tilde{s}_1 \rangle, \langle u_2, \tilde{s}_2 \rangle, \cdots, \langle u_n, \tilde{s}_n \rangle \right) \otimes \tilde{s} \tag{37}$$

Theorem 3.24. Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i} \right]$ ($i = 1, 2, \cdots, n$) be a collection of uncertain linguistic variables, if $r > 0$, then

$$\mathrm{IULOWG}\left(\langle u_1, \tilde{s}_1^r \rangle, \langle u_2, \tilde{s}_2^r \rangle, \cdots, \langle u_n, \tilde{s}_n^r \rangle \right) = \left(\mathrm{IULOWG}\left(\langle u_1, \tilde{s}_1 \rangle, \langle u_2, \tilde{s}_2 \rangle, \cdots, \langle u_n, \tilde{s}_n \rangle \right) \right)^r \tag{38}$$

Theorem 3.25. Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i} \right]$ ($i = 1, 2, \cdots, n$) be a collection of uncertain linguistic variables, if $r > 0$ and $\tilde{s} = \left[s_\alpha, s_\beta \right]$ is an uncertain linguistic variable, then

$$\mathrm{IULOWG}\left(\langle u_1, \tilde{s}_1^r \otimes \tilde{s} \rangle, \langle u_2, \tilde{s}_2^r \otimes \tilde{s} \rangle, \cdots, \langle u_n, \tilde{s}_n^r \otimes \tilde{s} \rangle \right) = \left(\mathrm{IULOWG}\left(\langle u_1, \tilde{s}_1 \rangle, \langle u_2, \tilde{s}_2 \rangle, \cdots, \langle u_n, \tilde{s}_n \rangle \right) \right)^r \otimes \tilde{s} \tag{39}$$

Theorem 3.26. Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i} \right]$ and $\tilde{s}'_i = \left[s_{\alpha'_i}, s_{\beta'_i} \right]$ ($i = 1, 2, \cdots, n$) be two collections of uncertain linguistic variables, then

$$\begin{aligned} &\mathrm{IULOWG}\left(\langle u_1, \tilde{s}_1 \otimes \tilde{s}'_1 \rangle, \langle u_2, \tilde{s}_2 \otimes \tilde{s}'_2 \rangle, \cdots, \langle u_n, \tilde{s}_n \otimes \tilde{s}'_n \rangle \right) \\ &= \mathrm{IULOWG}\left(\langle u_1, \tilde{s}_1 \rangle, \langle u_2, \tilde{s}_2 \rangle, \cdots, \langle u_n, \tilde{s}_n \rangle \right) \otimes \mathrm{IULOWG}\left(\langle u_1, \tilde{s}'_1 \rangle, \langle u_2, \tilde{s}'_2 \rangle, \cdots, \langle u_n, \tilde{s}'_n \rangle \right) \end{aligned} \tag{40}$$

In what follows, I will investigate the relationship between the IGULOWA operator and the IGULOWG operator.

Theorem 3.27. Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i} \right]$ ($i = 1, 2, \cdots, n$) be a collection of uncertain linguistic variables, then we have

(1) $\mathrm{IGULOWA}_\lambda \left(\mathrm{neg}(\tilde{s}_1), \mathrm{neg}(\tilde{s}_2), \cdots, \mathrm{neg}(\tilde{s}_n) \right) = \mathrm{neg}\left(\mathrm{IGULOWG}_\lambda \left(\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_n \right) \right)$ (41)

(2) $\mathrm{IGULOWG}_\lambda \left(\mathrm{neg}(\tilde{s}_1), \mathrm{neg}(\tilde{s}_2), \cdots, \mathrm{neg}(\tilde{s}_n) \right) = \mathrm{neg}\left(\mathrm{IGULOWA}_\lambda \left(\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_n \right) \right)$ (42)

Proof. (1) According to Definition 2.1, Eq. (9), and Eq. (31), we can get

$$\mathrm{IGULOWA}_{\lambda}\left(\mathrm{neg}(\tilde{s}_1),\mathrm{neg}(\tilde{s}_2),\cdots,\mathrm{neg}(\tilde{s}_n)\right)$$

$$=\left[s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(t-\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)},s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(t-\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)}\right]$$

$$=\left[s_{t-f^{-1}\left(\frac{1}{\lambda}f\left(g^{-1}\left(\sum_{j=1}^{n}w_j g\left(f^{-1}\left(\lambda f\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)},s_{t-f^{-1}\left(\frac{1}{\lambda}f\left(g^{-1}\left(\sum_{j=1}^{n}w_j g\left(f^{-1}\left(\lambda f\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)}\right]$$

$$=\mathrm{neg}\left(\left[s_{f^{-1}\left(\frac{1}{\lambda}f\left(g^{-1}\left(\sum_{j=1}^{n}w_j g\left(f^{-1}\left(\lambda f\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)},s_{f^{-1}\left(\frac{1}{\lambda}f\left(g^{-1}\left(\sum_{j=1}^{n}w_j g\left(f^{-1}\left(\lambda f\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)}\right]\right)$$

$$=\mathrm{neg}\left(\mathrm{IGULOWG}_{\lambda}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)\right)$$

(2) According to Definition 2.1, Eq. (9), and Eq. (31), we have

$$\mathrm{IGULOWG}_{\lambda}\left(\mathrm{neg}(\tilde{s}_1),\mathrm{neg}(\tilde{s}_2),\cdots,\mathrm{neg}(\tilde{s}_n)\right)$$

$$=\left[s_{f^{-1}\left(\frac{1}{\lambda}f\left(g^{-1}\left(\sum_{j=1}^{n}w_j g\left(f^{-1}\left(\lambda f\left(t-\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)},s_{f^{-1}\left(\frac{1}{\lambda}f\left(g^{-1}\left(\sum_{j=1}^{n}w_j g\left(f^{-1}\left(\lambda f\left(t-\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)}\right]$$

$$=\left[s_{t-g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)},s_{t-g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)}\right]$$

$$=\mathrm{neg}\left(\left[s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)},s_{g^{-1}\left(\frac{1}{\lambda}g\left(f^{-1}\left(\sum_{j=1}^{n}w_j f\left(g^{-1}\left(\lambda g\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)}\right]\right)$$

$$=\mathrm{neg}\left(\mathrm{IGULOWA}_{\lambda}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)\right)$$

This completes the proof of Theorem 3.27. □

We next look at some special cases of the IGULOWG operator obtained by using different choices of the parameters: the associated weighting vector w, the order inducing variable u_j, and the parameter λ.

(1) If $\lambda=1$, then the IGULOWG operator reduces to the following formula:

$$\mathrm{IULOWG}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)=\left[s_{g^{-1}\left(\sum_{j=1}^{n}w_j g\left(\alpha_{\sigma(j)}\right)\right)},s_{g^{-1}\left(\sum_{j=1}^{n}w_j g\left(\beta_{\sigma(j)}\right)\right)}\right] \quad (43)$$

which is called an induced uncertain linguistic ordered weighted geometric (IULOWG) operator, where $\tilde{s}_{\sigma(j)}$ is presented as in Definition 3.4.

(2) If $\lambda=-1$, then the IGULOWG operator reduces to the following formula:

$$\mathrm{IULOWHG}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)=\left[s_{f^{-1}\left(-f\left(g^{-1}\left(\sum_{j=1}^{n}w_j g\left(f^{-1}\left(-f\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)},s_{f^{-1}\left(-f\left(g^{-1}\left(\sum_{j=1}^{n}w_j g\left(f^{-1}\left(-f\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)}\right] \quad (44)$$

which is called an induced uncertain linguistic ordered weighted harmonic geometric (IULOWHG) operator.

(3) If $\lambda=2$, then the IGULOWG operator reduces to the following formula:

$$\mathrm{IULOWQG}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)=\left[s_{f^{-1}\left(\frac{1}{2}f\left(g^{-1}\left(\sum_{j=1}^{n}w_j g\left(f^{-1}\left(2f\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)},s_{f^{-1}\left(\frac{1}{2}f\left(g^{-1}\left(\sum_{j=1}^{n}w_j g\left(f^{-1}\left(2f\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)}\right] \quad (45)$$

which is called an induced uncertain linguistic ordered weighted quadratic geometric (IULOWQG) operator.

(4) If $u_j=\tilde{s}_j$ for all j, then the IGULOWG operator reduces to the following formula:

$$\mathrm{GULOWG}_{\lambda}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)=\left[s_{f^{-1}\left(\frac{1}{\lambda}f\left(g^{-1}\left(\sum_{j=1}^{n}w_j g\left(f^{-1}\left(\lambda f\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)},s_{f^{-1}\left(\frac{1}{\lambda}f\left(g^{-1}\left(\sum_{j=1}^{n}w_j g\left(f^{-1}\left(\lambda f\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)}\right] \quad (46)$$

which is called an generalized uncertain linguistic ordered weighted geometric (GULOWG) operator, where $\tilde{s}_{\sigma(j)}$ is the *j*th largest of the $\tilde{s}_i$.

(5) If $u_j=\tilde{s}_j$ for all j and $\lambda=1$, then the IGULOWG operator reduces to the following formula:

$$\mathrm{ULOWG}\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)=\left[s_{g^{-1}\left(\sum_{j=1}^{n}w_j g\left(\alpha_{\sigma(j)}\right)\right)},s_{g^{-1}\left(\sum_{j=1}^{n}w_j g\left(\beta_{\sigma(j)}\right)\right)}\right] \quad (47)$$

which is called an uncertain linguistic ordered weighted geometric (ULOWG) operator, where $\tilde{s}_{\sigma(j)}$ is the *j*th largest of the $\tilde{s}_i$.

(6) If $u_j = -j$ for all j and the associated weighting vector $w = \left(w_1, w_2, \cdots, w_n\right)^T$ is the weight vector of $\tilde{s}_j$ ($j = 1, 2, \cdots, n$), then the IGULOWG operator reduces to the following formula:

$$\mathrm{GULWG}_{\lambda}\left(\left\langle u_1, \tilde{s}_1\right\rangle, \left\langle u_2, \tilde{s}_2\right\rangle, \cdots, \left\langle u_n, \tilde{s}_n\right\rangle\right) = \left[s_{f^{-1}\left(\frac{1}{\lambda} f\left(g^{-1}\left(\sum_{j=1}^{n} w_j g\left(f^{-1}\left(\lambda f\left(\alpha_j\right)\right)\right)\right)\right)\right)}, s_{f^{-1}\left(\frac{1}{\lambda} f\left(g^{-1}\left(\sum_{j=1}^{n} w_j g\left(f^{-1}\left(\lambda f\left(\beta_j\right)\right)\right)\right)\right)\right)} \right] \quad (48)$$

which is called an generalized uncertain linguistic weighted geometric (GULWG) operator.

(7) If $u_j = -j$ for all j, the associated weighting vector $w = \left(w_1, w_2, \cdots, w_n\right)^T$ is the weight vector of $\tilde{s}_j$ ($j = 1, 2, \cdots, n$), and $\lambda = 1$, then the IGULOWG operator reduces to the following formula:

$$\mathrm{ULWG}\left(\left\langle u_1, \tilde{s}_1\right\rangle, \left\langle u_2, \tilde{s}_2\right\rangle, \cdots, \left\langle u_n, \tilde{s}_n\right\rangle\right) = \left[s_{g^{-1}\left(\sum_{j=1}^{n} w_j g\left(\alpha_j\right)\right)}, s_{g^{-1}\left(\sum_{j=1}^{n} w_j g\left(\beta_j\right)\right)} \right] \quad (49)$$

which is called an uncertain linguistic weighted geometric (ULWG) operator.

(8) If $w = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^T$, then the IGULOWG operator reduces to the following formula:

$$\mathrm{IGULOG}_{\lambda}\left(\left\langle u_1, \tilde{s}_1\right\rangle, \left\langle u_2, \tilde{s}_2\right\rangle, \cdots, \left\langle u_n, \tilde{s}_n\right\rangle\right) = \left[s_{f^{-1}\left(\frac{1}{\lambda} f\left(g^{-1}\left(\sum_{j=1}^{n} \frac{1}{n} g\left(f^{-1}\left(\lambda f\left(\alpha_{\sigma(j)}\right)\right)\right)\right)\right)\right)}, s_{f^{-1}\left(\frac{1}{\lambda} f\left(g^{-1}\left(\sum_{j=1}^{n} \frac{1}{n} g\left(f^{-1}\left(\lambda f\left(\beta_{\sigma(j)}\right)\right)\right)\right)\right)\right)} \right] \quad (50)$$

which is called an induced generalized uncertain linguistic ordered geometric (IGULOG) operator.

(9) If $w = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^T$ and $\lambda = 1$, then the IGULOWG operator reduces to the following formula:

$$\mathrm{IULOG}\left(\left\langle u_1, \tilde{s}_1\right\rangle, \left\langle u_2, \tilde{s}_2\right\rangle, \cdots, \left\langle u_n, \tilde{s}_n\right\rangle\right) = \left[s_{g^{-1}\left(\sum_{j=1}^{n} \frac{1}{n} g\left(\alpha_{\sigma(j)}\right)\right)}, s_{g^{-1}\left(\sum_{j=1}^{n} \frac{1}{n} g\left(\beta_{\sigma(j)}\right)\right)} \right] \quad (51)$$

which is called an induced uncertain linguistic ordered geometric (IULOG) operator.

4. AN APPROACH BASED ON THE DEVELOPED OPERATORS TO GROUP DECISION MAKING WITH UNCERTAIN LINGUISTIC INFORMATION

For a multiple attribute group decision making (MAGDM) with uncertain linguistic information, let $X=\{x_1,x_2,\cdots,x_m\}$ be a set of m alternatives, $C=\{c_1,c_2,\ldots,c_n\}$ be a collection of n attributes, whose weight vector is $w=(w_1,w_2,\cdots,w_n)^T$, with $w_j\in[0,1]$, $j=1,2,\cdots,n$, and $\sum_{j=1}^{n} w_j=1$, where w_j denotes the importance degree of the attribute c_j, and let $D=\{d_1,d_2,\cdots,d_l\}$ be a set of l decision makers, whose weight vector is $\omega=(\omega_1,\omega_2,\cdots,\omega_l)^T$ with $\omega_k\in[0,1]$, $k=1,2,\cdots,l$, and $\sum_{k=1}^{l}\omega_k=1$, where ω_k denotes the importance degree of the decision maker d_k. Each decision maker provides his/her own uncertain linguistic decision matrix $\tilde{A}^{(k)}=\left(\tilde{a}_{ij}^{(k)}\right)_{m\times n}$ ($k=1,2,\cdots,l$) as follows:

$$\tilde{A}^{(k)}=\left(\tilde{a}_{ij}^{(k)}\right)_{m\times n}=\begin{bmatrix} \tilde{a}_{11}^{(k)} & \tilde{a}_{12}^{(k)} & \cdots & \tilde{a}_{1n}^{(k)} \\ \tilde{a}_{21}^{(k)} & \tilde{a}_{22}^{(k)} & \cdots & \tilde{a}_{2n}^{(k)} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{a}_{m1}^{(k)} & \tilde{a}_{m2}^{(k)} & \cdots & \tilde{a}_{mn}^{(k)} \end{bmatrix}$$

where $\tilde{a}_{ij}^{(k)}\in\tilde{S}$ is a performance value, which takes the form of uncertain linguistic variable, given by the decision maker $d_k\in D$, for the alternative $x_i\in X$ with respect to the attribute $c_j\in C$.

In the following, based on the proposed operators, we develop an approach to group decision making with uncertain linguistic information, which includes the following steps:

Step 1. Utilize the IGULOWA operator (Eq. (9))

$$\tilde{a}_{ij}=\mathrm{IGULOWA}_{\lambda}\left(\left\langle\omega_1,\tilde{a}_{ij}^{(1)}\right\rangle,\left\langle\omega_2,\tilde{a}_{ij}^{(2)}\right\rangle,\cdots,\left\langle\omega_l,\tilde{a}_{ij}^{(l)}\right\rangle\right)=\left(\bigoplus_{k=1}^{l}\left(\xi_k\left(\tilde{a}_{ij}^{(\sigma(k))}\right)^{\lambda}\right)\right)^{1/\lambda},\ i=1,2,\cdots,m;\ j=1,2,\cdots,n \tag{52}$$

or the IGULOWG operator (Eq. (31))

$$\tilde{a}_{ij}=\mathrm{IGULOWG}_{\lambda}\left(\left\langle\xi_1,\tilde{a}_{ij}^{(1)}\right\rangle,\left\langle\xi_2,\tilde{a}_{ij}^{(2)}\right\rangle,\cdots,\left\langle\xi_l,\tilde{a}_{ij}^{(l)}\right\rangle\right)=\frac{1}{\lambda}\left(\bigotimes_{k=1}^{l}\left(\lambda\tilde{a}_{ij}^{(\sigma(k))}\right)^{\omega_k}\right),\ i=1,2,\cdots,m;\ j=1,2,\cdots,n \tag{53}$$

to aggregate all the uncertain linguistic decision matrices $\tilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{m\times n}$ ($k = 1,2,\cdots,l$) into a collective uncertain linguistic decision matrix $\tilde{A} = \left(\tilde{a}_{ij}\right)_{m\times n}$, where $\lambda \in (0,+\infty)$, $\xi = (\xi_1,\xi_2,\cdots,\xi_l)^T$ ($\xi_k \in [0,1]$, $k = 1,2,\cdots,l$, and $\sum_{k=1}^{l}\xi_k = 1$) is the order inducing variable, and $\omega = (\omega_1,\omega_2,\cdots,\omega_l)^T$ is the weight vector of decision makers.

Step 2. Utilize the decision information given in matrix $\tilde{A}$, and the GULWA operator (Eq. (25))

$$\tilde{a}_i = \mathrm{GULWA}_\lambda\left(\tilde{a}_{i1},\tilde{a}_{i2},\cdots,\tilde{a}_{in}\right) = \left(\underset{j=1}{\overset{n}{\oplus}}\left(w_j\tilde{a}_{ij}^{\lambda}\right)\right)^{1/\lambda}, \quad i = 1,2,\cdots,m \tag{54}$$

or the GULWG operator (Eq. (48))

$$\tilde{a}_i = \mathrm{GULWG}_\lambda\left(\tilde{a}_{i1},\tilde{a}_{i2},\cdots,\tilde{a}_{in}\right) = \frac{1}{\lambda}\left(\underset{j=1}{\overset{n}{\otimes}}\left(\lambda\tilde{a}_{ij}\right)^{w_j}\right), \quad i = 1,2,\cdots,m \tag{55}$$

to derive the collective overall preference value $\tilde{a}_i$ of the alternative x_i , where $w = (w_1,w_2,\cdots,w_n)^T$ is the weight vector of attributes.

Step 3. Calculate the scores $s(\tilde{a}_i)$ ($i = 1,2,\cdots,m$) of the collective overall preference values $\tilde{a}_i$ ($i = 1,2,\cdots,m$) by Definition 2.3.

Step 4. Rank all the alternatives x_i ($i = 1,2,\cdots,m$) and select the best one(s) in accordance with the collective overall preference values $\tilde{a}_i$ ($i = 1,2,\cdots,m$).

Step 5. End.

5. NUMERICAL EXAMPLE

In the following, we use a numerical example adapted from [4] to illustrate our approach.

Example 5.1 [4]. Let us suppose a company which is planning his production strategy for the next year and wants to introduce a new product but still do not know the type of customers to address. A panel is provided with five possible alternatives:

(1) x_1 is a luxury product oriented to the very rich customers;
(2) x_2 is a luxury product oriented to the rich customers;
(3) x_3 is a product oriented to the average customers;
(4) x_4 is a product oriented to the poor customers.

The company must make a decision according to the following four attributes (suppose that the weight vector of four attributes is $w = (0.3,0.1,0.25,0.35)^T$):

(1) c_1 is the risk analysis;
(2) c_2 is the growth analysis;
(3) c_3 is the social-political impact analysis; and
(4) c_4 is the environmental impact analysis.

The four possible alternatives x_i $(i=1,2,3,4)$ are to be evaluated using the linguistic term set

$$S=\left\{\begin{array}{l}s_0=\text{extremely poor},\ s_1=\text{very poor},\ s_2=\text{poor},\ s_3=\text{slightly poor},\ s_4=\text{fair},\\ s_5=\text{slightly good},\ s_6=\text{good},\ s_7=\text{very good},\ s_8=\text{extremely good}\end{array}\right\}$$

by three decision makers d_k $(k=1,2,3)$ (suppose that the weight vector of three decision makers is $\omega=(0.1,0.6,0.3)^T$) under the above four attributes, and construct, respectively, the uncertain linguistic decision matrices $\tilde{A}^{(k)}=\left(\tilde{a}_{ij}^{(k)}\right)_{5\times4}$ $(k=1,2,3)$ as shown in Tables 1-3.

The following steps are used to get the best alternative(s):

Table 1. Uncertain linguistic decision matrix $\tilde{A}^{(1)}$ provided by d_1

1	c_1	c_2	c_3	c_4
x_1	$[s_4,s_6]$	$[s_5,s_6]$	$[s_1,s_2]$	$[s_2,s_3]$
x_2	$[s_2,s_3]$	$[s_5,s_7]$	$[s_1,s_2]$	$[s_1,s_3]$
x_3	$[s_1,s_2]$	$[s_5,s_6]$	$[s_5,s_7]$	$[s_3,s_4]$
x_4	$[s_3,s_3]$	$[s_3,s_4]$	$[s_4,s_5]$	$[s_5,s_6]$
x_5	$[s_1,s_3]$	$[s_5,s_7]$	$[s_5,s_6]$	$[s_2,s_3]$

Table 2. Uncertain linguistic decision matrix $\tilde{A}^{(2)}$ provided by d_2

2	c_1	c_2	c_3	c_4
x_1	$[s_5,s_6]$	$[s_5,s_6]$	$[s_1,s_3]$	$[s_3,s_4]$
x_2	$[s_1,s_3]$	$[s_4,s_6]$	$[s_1,s_2]$	$[s_5,s_7]$
x_3	$[s_1,s_2]$	$[s_6,s_7]$	$[s_5,s_6]$	$[s_3,s_4]$
x_4	$[s_5,s_6]$	$[s_4,s_5]$	$[s_1,s_2]$	$[s_5,s_6]$
x_5	$[s_3,s_5]$	$[s_1,s_3]$	$[s_1,s_2]$	$[s_6,s_7]$

Step 1. Utilize the IGULOWA operator (suppose that $\xi=(0.2,0.5,0.3)^T$, $g(x)=-\log\left(\frac{x}{8}\right)$, and $\lambda=2$) (Eq. (52)) to aggregate all the uncertain linguistic decision matrices $\tilde{A}^{(k)}=\left(\tilde{a}_{ij}^{(k)}\right)_{5\times4}$ ($k=1,2,3$) into the collective uncertain linguistic decision matrix $\tilde{A}=\left(\tilde{a}_{ij}\right)_{5\times4}$ (see Table 4).

Table 3. Uncertain linguistic decision matrix $\tilde{A}^{(3)}$ provided by d_3

3	c_1	c_2	c_3	c_4
x_1	$[s_3, s_4]$	$[s_5, s_7]$	$[s_3, s_4]$	$[s_3, s_4]$
x_2	$[s_3, s_6]$	$[s_1, s_2]$	$[s_1, s_2]$	$[s_6, s_7]$
x_3	$[s_2, s_3]$	$[s_2, s_3]$	$[s_5, s_7]$	$[s_3, s_5]$
x_4	$[s_4, s_5]$	$[s_1, s_2]$	$[s_1, s_2]$	$[s_3, s_4]$
x_5	$[s_3, s_4]$	$[s_4, s_5]$	$[s_3, s_5]$	$[s_6, s_7]$

Table 4. Collective uncertain linguistic decision matrix $\tilde{A}$

4	c_1	c_2	c_3	c_4
x_1	$[s_{3.8352}, s_{5.2287}]$	$[s_{5.0000}, s_{6.5959}]$	$[s_{2.2662}, s_{3.3556}]$	$[s_{2.7469}, s_{3.7415}]$
x_2	$[s_{2.4477}, s_{4.9757}]$	$[s_{3.5114}, s_{5.4783}]$	$[s_{1.0000}, s_{2.0000}]$	$[s_{5.0834}, s_{6.4692}]$
x_3	$[s_{1.5869}, s_{2.5602}]$	$[s_{4.3276}, s_{5.4187}]$	$[s_{5.0000}, s_{6.8561}]$	$[s_{3.0000}, s_{4.5534}]$
x_4	$[s_{4.0041}, s_{4.8514}]$	$[s_{2.5917}, s_{3.5380}]$	$[s_{2.4363}, s_{3.3577}]$	$[s_{4.2055}, s_{5.2287}]$
x_5	$[s_{2.5909}, s_{4.0041}]$	$[s_{4.0470}, s_{5.7128}]$	$[s_{3.6088}, s_{5.0511}]$	$[s_{5.3672}, s_{6.4692}]$

Step 2. Utilize the decision information given in matrix $\tilde{A} = \left(\tilde{a}_{ij}\right)_{m\times n}$, and the GULWA operator (suppose that $g(x) = -\log\left(\frac{x}{8}\right)$ and $\lambda = 2$) (Eq. (25)) to derive the collective overall preference value $\tilde{a}_i$ ($i = 1,2,3,4,5$) of the alternative x_i ($i = 1,2,3,4,5$):

$\tilde{a}_1 = [s_{3.3492}, s_{4.6661}]$, $\tilde{a}_2 = [s_{3.6667}, s_{5.3655}]$, $\tilde{a}_3 = [s_{3.5805}, s_{5.2467}]$, $\tilde{a}_4 = [s_{3.6620}, s_{4.6063}]$, $\tilde{a}_5 = [s_{4.2359}, s_{5.5498}]$

Step 3. Calculate the scores $s(\tilde{a}_i)$ ($i = 1,2,3,4,5$) of the collective overall preference values $\tilde{a}_i$ ($i = 1,2,3,4,5$) by Definition 2.3:

$$s(\tilde{a}_1) = 0.5010,\ s(\tilde{a}_2) = 0.5645,\ s(\tilde{a}_3) = 0.5517,\ s(\tilde{a}_4) = 0.5168,\ s(\tilde{a}_5) = 0.6116$$

Step 4. Rank all the alternatives x_i ($i = 1,2,3,4,5$) in a descending order by Definition 2.5:

$$x_5 \succ x_2 \succ x_3 \succ x_4 \succ x_1$$

thus the optimal alternative is x_5.

In the following, we will analyze how different values of the parameter λ change the aggregation results. As λ is assigned different values between 0 and 10, the score functions of the alternatives obtained by the IGULOWA operator are shown in Fig. 1.

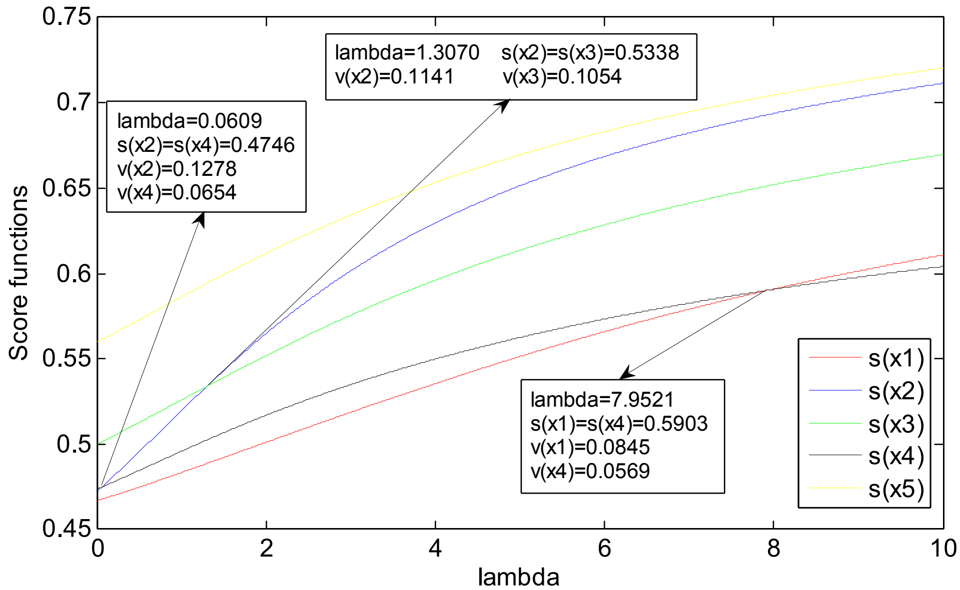

Fig. 1. Score functions for alternatives obtained by the IGULOWA operator

Fig. 1 demonstrates that all the score functions increase as λ increases from 0 to 10, from which we can find that

(1) when $\lambda \in (0, 0.0609]$, the ranking of the four alternatives is $x_5 \succ x_3 \succ x_4 \succ x_2 \succ x_1$ and the best choice is x_5.
(2) when $\lambda \in (0.0609, 1.3070]$, the ranking of the four alternatives is $x_5 \succ x_3 \succ x_2 \succ x_4 \succ x_1$ and the best choice is x_5.
(3) when $\lambda \in (1.3070,\ 7.9521]$, the ranking of the four alternatives is $x_5 \succ x_2 \succ x_3 \succ x_4 \succ x_1$ and the best choice is x_5.
(4) when $\lambda \in (7.9521, 10]$, the ranking of the four alternatives is $x_5 \succ x_2 \succ x_3 \succ x_1 \succ x_4$ and the best choice is x_5.

In the above example, if we use the IGULOWG operator instead of the IGULOWA operator to aggregate the values of the alternatives, then the score functions of the alternatives are shown in Fig. 2. From Fig. 2, we can see that all the score functions obtained by the IGULOWG operator decrease as the parameter λ increases from 0 to 10 and the aggregation arguments are kept fixed. From Fig. 2, we can find that

(1) when $\lambda \in (0, 1.2208)$, the ranking of the four alternatives is $x_5 \succ x_3 \succ x_1 \succ x_4 \succ x_2$ and the best choice is x_5.
(2) when $\lambda \in [1.2208, 10]$, the ranking of the four alternatives is $x_5 \succ x_1 \succ x_3 \succ x_4 \succ x_2$ and the best choice is x_5.

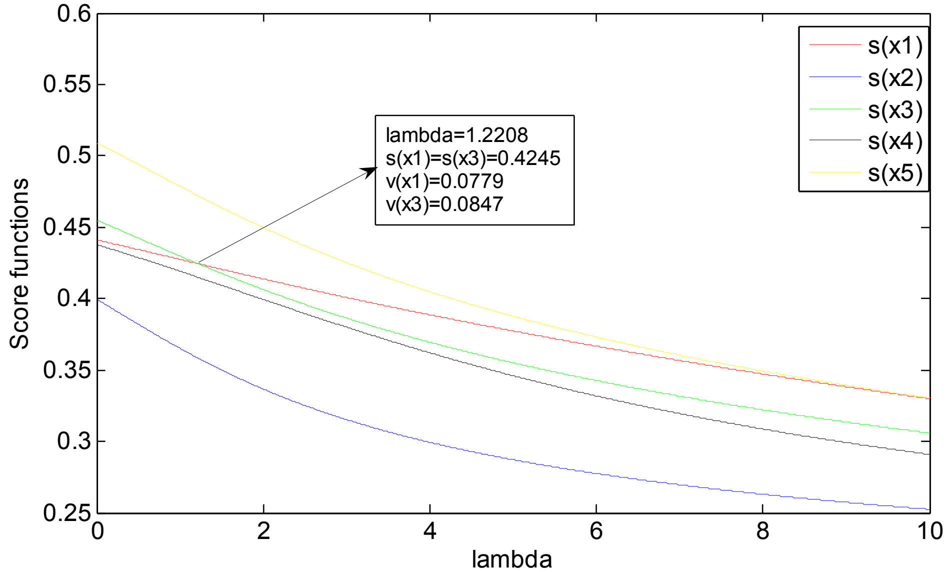

Fig. 2. Score functions for alternatives obtained by the IGULOWG operator.

Fig. 3 illustrates the deviation values between the score functions obtained by the IGULOWA operator and the ones obtained by the IGULOWG operator, from which we can find that the values obtained by the IGULOWA operator are greater than the ones obtained by the IGULOWG operator for the same value of the parameter λ and the same aggregation values, and the deviation values increase as the value of the parameter λ increases.

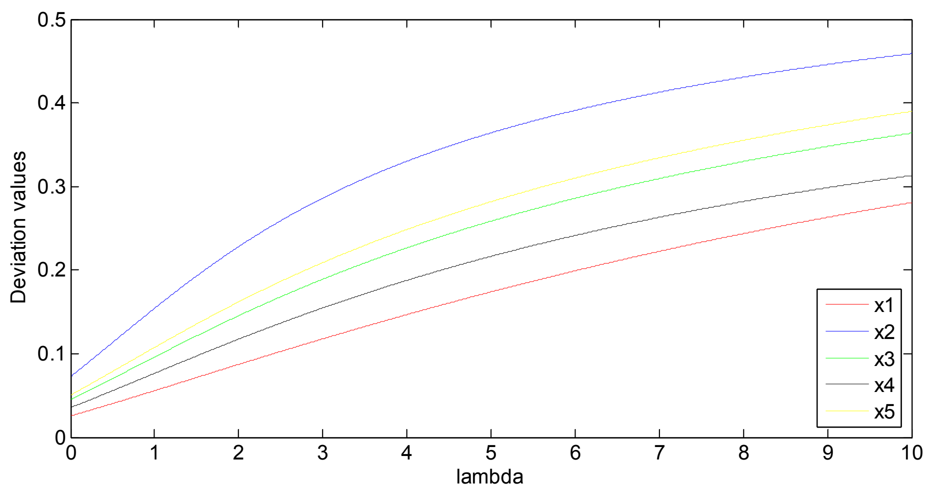

Fig. 3. Deviation values for alternatives between the IGULOWA and IGULOWG operators.

Fig. 3 indicates that the IGULOWA operator can obtain more favorable (or optimistic) expectations, and therefore can be considered as an optimistic operator, while the IGULOWG operator has more unfavorable (or pessimistic) expectations, and therefore can be considered as a pessimistic operator. The values of the parameter λ can be considered as the optimistic or pessimistic levels. According to Figs. 1, 2, and 3, we can conclude that the decision makers who take a gloomy view of the prospects could use the IGULOWG operator and choose the bigger values of the parameter λ, while the decision makers who are optimistic could use the IGULOWA operator and choose the bigger values of the parameter λ.

6. COMPARISON WITH PREVIOUSLY PROPOSED UNCERTAIN LINGUISTIC AGGREGATION OPERATORS AND MAGDM METHODS

In this section, we perform a comparison analysis between our new operators and approach and other uncertain linguistic aggregation operators and MAGDM methods, and then highlight the advantages of the new operators and approach.

6.1 The Uncertain Linguistic Aggregation Operators and Approach Proposed by Xu [16]

We begin our comparison by using the operations and method developed by Xu [16] to address Example 5.1. First, we review the IULOWA and ULWA operators developed by Xu [16] as follows:

$$\mathrm{IULOWA}_w\left(\langle u_1,\tilde{s}_1\rangle,\langle u_2,\tilde{s}_2\rangle,\cdots,\langle u_n,\tilde{s}_n\rangle\right)=\xi_1\tilde{s}_{\sigma(1)}\oplus\xi_2\tilde{s}_{\sigma(2)}\oplus\cdots\oplus\xi_n\tilde{s}_{\sigma(n)}=\left[s_{\sum_{j=1}^{n}\xi_j\alpha_{\sigma(j)}},s_{\sum_{j=1}^{n}\xi_j\beta_{\sigma(j)}}\right] \tag{56}$$

where $\xi=\left(\xi_1,\xi_2,\cdots,\xi_n\right)^T$ is a weighting vector, such that $\xi_j\in\left[0,1\right]$, $\sum_{j=1}^{n}\xi_j=1$, $\tilde{s}_{\sigma(j)}=\left[s_{\alpha_{\sigma(j)}},s_{\beta_{\sigma(j)}}\right]$ is the $\tilde{s}_i$ value of the ULOWA pair $\langle u_i,\tilde{s}_i\rangle$ having the *j*th largest u_i, and u_i in $\langle u_i,\tilde{s}_i\rangle$ is referred to as the order inducing variable and $\tilde{s}_i$ as the uncertain linguistic argument variable.

$$\mathrm{ULWA}_w\left(\tilde{s}_1,\tilde{s}_2,\cdots,\tilde{s}_n\right)=w_1\tilde{s}_1\oplus w_2\tilde{s}_2\oplus\cdots\oplus w_n\tilde{s}_n==\left[s_{\sum_{j=1}^{n}w_j\alpha_j},s_{\sum_{j=1}^{n}w_j\beta_j}\right] \tag{57}$$

Based on the IULOWA and ULWA operators [16], the following steps are involved to obtain the most desirable alternative(s):

Step 1. Use the IULOWA operator (Eq. (56)) (in order to be consistent with Example 5.1, suppose that $\xi=\left(0.2,0.5,0.3\right)^T$) to aggregate all the individual uncertain linguistic decision matrices $\tilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{5\times 4}$ ($k = 1,2,3$) into the collective uncertain linguistic decision matrix $\tilde{A} = \left(\tilde{a}_{ij}\right)_{5\times 4}$ (see Table 5).

Table 5. The collective uncertain linguistic decision matrix $\tilde{A}$

5	c_1	c_2	c_3	c_4
x_1	$[s_{3.7}, s_{5.0}]$	$[s_{5.0}, s_{6.5}]$	$[s_{2.0}, s_{3.2}]$	$[s_{2.7}, s_{3.7}]$
x_2	$[s_{2.3}, s_{4.5}]$	$[s_{2.8}, s_{4.3}]$	$[s_{1.0}, s_{2.0}]$	$[s_{4.3}, s_{5.8}]$
x_3	$[s_{1.5}, s_{2.5}]$	$[s_{3.7}, s_{4.7}]$	$[s_{5.0}, s_{6.8}]$	$[s_{3.0}, s_{4.5}]$
x_4	$[s_{3.9}, s_{4.6}]$	$[s_{2.2}, s_{3.2}]$	$[s_{1.9}, s_{2.9}]$	$[s_{4.0}, s_{5.0}]$
x_5	$[s_{2.4}, s_{3.9}]$	$[s_{3.7}, s_{5.2}]$	$[s_{3.2}, s_{4.7}]$	$[s_{4.8}, s_{5.8}]$

Step 2. Utilize the decision information given in matrix $\tilde{A}$, and the ULWA operator (Eq. (57)) (in order to be consistent with Example 5.1, suppose that $w = (0.3, 0.1, 0.25, 0.35)^T$) to derive the collective overall preference value $\tilde{a}_i$ of the alternative x_i:

$\tilde{a}_1 = [s_{3.0550}, s_{4.2450}]$, $\tilde{a}_2 = [s_{2.7250}, s_{4.3100}]$, $\tilde{a}_3 = [s_{3.1200}, s_{4.4950}]$, $\tilde{a}_4 = [s_{3.2650}, s_{4.1750}]$, $\tilde{a}_5 = [s_{3.5700}, s_{4.8950}]$

Step 3. Calculate the scores $s(\tilde{a}_i)$ ($i = 1,2,3,4,5$) of the collective overall preference values $\tilde{a}_i$ ($i = 1,2,3,4,5$) by Definition 2.3:

$$s(\tilde{a}_1) = 0.4563,\ s(\tilde{a}_2) = 0.4397,\ s(\tilde{a}_3) = 0.4759,\ s(\tilde{a}_4) = 0.4650,\ s(\tilde{a}_5) = 0.5291$$

Step 4. Rank all the alternatives x_i ($i = 1,2,3,4,5$) in a descending order by Definition 2.5:

$$x_5 \succ x_3 \succ x_4 \succ x_1 \succ x_2$$

thus the optimal alternative is x_5.

6.2 Discussion

It is easy to see that the optimal alternative obtained by the Xu' method [16] is the same as our method, which shows the effectiveness, preciseness, and reasonableness of our method. However, it is noticed that the ranking order of the alternatives obtained by our method is $x_5 \succ x_2 \succ x_3 \succ x_4 \succ x_1$, which is different from the ranking order obtained by the Xu' method [16]. Concretely, the ranking order between x_1, x_2, x_3, and x_4 obtained by two methods are just converse, i.e., $x_2 \succ x_3 \succ x_4 \succ x_1$ for our method while $x_3 \succ x_4 \succ x_1 \succ x_2$ for the Xu' method [16].

In the following, we will illustrate the advantages of the developed operators and approach over the existing uncertain linguistic aggregation operators and approaches:

(1) The existing linguistic aggregation operators and uncertain linguistic aggregation operators are constructed based on the Xu' addition and multiplication operations. As stated above, the Xu's addition and multiplication operations have some

drawbacks. As a result, the existing linguistic aggregation operators and uncertain linguistic aggregation operators are unreasonable. To overcome these drawbacks, we redefine the new addition and multiplication operations on uncertain linguistic variables, and based on which, we further develop several new uncertain linguistic aggregation operators. Thus, compared with the existing linguistic and uncertain linguistic aggregation operators, the newly developed aggregation operators are much precise and practical, thereby making the decision making more reasonable and reliable.

(2) The developed uncertain linguistic aggregation operators have some new desired properties, such as Theorems 3.13, 3.14, 3.15, 3.16, 3.23, 3.24, 3.25, 3.26, and 3.27, while the existing linguistic aggregation operations and uncertain linguistic aggregation operators do not have these properties.

(3) Ref. [16] mainly focuses on the IULOWA operators, neglecting the IULOWG operators. This paper investigates not only the IGULOWA operator but also the IGULOWG operator (see Subsection 3.3). Moreover, the relationship between the IGULOWA and IGULOWG operators is investigated in Theorem 3.27 and is illustrated in Fig. 3. The newly developed IGULOWA operator adds to the IULOWA operator an additional parameter controlling the power to which the argument values are raised. When we use different choices for the parameter λ, we obtain some special cases. The IGULOWG operator is a combination of the IGULOWA operator and the geometric mean. The IGULOWA operator is based on the arithmetic average, which focuses on the group decision but neglects the importance of the individual decisions. The IGULOWG operator is based on the geometric mean, which focuses on individual decisions but may neglect the importance of the group decision.

Furthermore, we investigate the variation trend of the developed aggregation operators with respect to the parameter λ (see Figs. 1 and 2) and examine the intersections of lines in the figures, which can indicate the role of the parameter λ in the ranking of the alternatives. In addition, considering that the selection of the appropriate values of the parameter λ for various situations is an interesting topic and is worthy of further study, we indicate how to select appropriate values of the parameter λ for various situations.

(4) Ref. [16] utilizes only the IULOWA operators to develop an approach for solving MAGDM problems with uncertain linguistic information, while our study utilizes the IGULOWA and IGULOWG operators to develop an approach for solving MAGDM problems with uncertain linguistic information. Furthermore, a comparative analysis of the decision results obtained by the IGULOWA and IGULOWG operators is shown in Example 5.1 (see Fig. 3).

(5) The new operators are constructed on the basis of the extended t-conorm and t-norm, which are generated by an additive function $g(x)$ and its dual function $f(x)=g(t-x)$. When the additive generator $g(x)$ is assigned different forms, we can obtain some specific extended t-conorms and t-norms, and then obtain some specific uncertain linguistic aggregation operators. The prominent characteristic of the developed operators is that they include a variety of uncertain linguistic aggregation operators when the additive generator $g(x)$ is assigned different forms.

7. CONCLUSION

In this paper, we have defined the extended triangular conorm and triangular norm in $[0,t]$ to deal with the linguistic information and uncertain linguistic information due to the fact that the existing operational laws of linguistic variables and uncertain linguistic variables have some drawbacks. Based on the extended triangular conorm and triangular norm, we have defined some new operational laws of linguistic variables and uncertain linguistic variables, studied their properties and correlations, and based on which, two uncertain linguistic aggregation operators, including the IGULOWA operator and IGULOWG operator, for the uncertain linguistic information have been presented. Based on the IGULOWA and the IGULOWG operators, we have proposed an approach to multiple attribute group decision making with uncertain linguistic information. Finally, a numerical example has been given to illustrate the developed approach using an investment problem.

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COMPETING INTERESTS

Author has declared that no competing interests exist.

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