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On a Weighted Retro Banach Frames for Discrete Signal Spaces

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Abstract

In this paper we introduce a weighted retro Banach frame for a discrete signal space. Necessary and sufficient condition for the existence of weighted retro Banach frames is obtained. Construction of weighted retro Banach frames from bounded linear operator is discussed. A Paley-Wiener type perturbation result for weighted retro Banach frame in Banach space setting is given.

Keywords: Frames; Banach frames; retro Banach frames; Perturbation.
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1 Introduction

Duffin and Schaeffer [1], introduced and studied frames for Hilbert spaces. Later, in 1986, Daubechies, Grossmann and Meyer [2] found new applications to wavelet and Gabor transforms in which frames played an important role. The basic theory of frames can be found in [3, 4, 5, 6, 7, 8] and references

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therein. A sequence $\{f_n\}$ in a separable Hilbert space $\mathbb{H}$ is called *frame* (Hilbert) for $\mathbb{H}$ if there exists positive constants A and B such that

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \text{ for all } f \in \mathbb{H}.$$

The operator $T : \ell^2 \to \mathbb{H}$ given by $T(\{c_n\}) = \sum\limits_{n=1}^{\infty} c_n f_n, \{c_n\} \in \ell^2$, is called the *synthesis operator* or *pre-frame operator*. Adjoint of T is given by $T^* : \mathbb{H} \to \ell^2$, $T^*(f) = \{\langle f, f_n\rangle\}$ and is called the *analysis operator*. Composing T and T^* we obtain the *frame operator* $S = TT^* : \mathbb{H} \to \mathbb{H}$ given by

$$S(f) = \sum_{n=1}^{\infty}\langle f, f_n\rangle f_n, f \in \mathbb{H}.$$

The frame operator S is a positive continuous invertible linear operator from $\mathbb{H}$ onto $\mathbb{H}$. Thus, each vector $f \in \mathbb{H}$ can be written as:

$$f = SS^{-1}f = \sum_{n=1}^{\infty}\langle S^{-1}f, f_n\rangle f_n. \tag{1.1}$$

Gröchenig in [9] generalized Hilbert frames to Banach frames. In this paper we introduce and study weighted retro Banach frames in Banach spaces. Necessary and/or sufficient conditions for a weighted retro Banach frame to be an exact are given. An application of weighted retro Banach frames is discussed.

Now we recall basic definitions and notations which is required throughout this paper. The set $\mathcal{X}$ denote a infinite dimensional Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\mathcal{X}^*$ the conjugate space of $\mathcal{X}$. For a sequence $\{f_n\} \subset \mathcal{X}^*$, $[f_n]$ denotes the closure of span$\{f_n\}$ in the norm topology of $\mathcal{X}^*$ and $\widetilde{[f_n]}$ the closure of span$\{f_n\}$ in $\sigma(\mathcal{X}^*, \mathcal{X})$-topology. A weight $\omega = \{\omega_n\}$ is a sequence of positive real numbers. The space of all bounded linear operators from a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.

Definition 1.1. ([10]). A system $(\{f_k\}, T)$ $(\{f_k\} \subset \mathcal{X}, T : \mathcal{Z}_d \to \mathcal{X}^*)$ is called a *retro Banach frame* for $\mathcal{X}^*$ with respect to an associated sequence space $\mathcal{Z}_d$ if,

(i) $\{f^*(f_k)\} \in \mathcal{Z}_d$, for each $f^* \in \mathcal{X}^*$;

(ii) there exist positive constants $(0 < \mathrm{A}_0 \leq \mathrm{B}_0 < \infty)$ such that

$$\mathrm{A}_0\|f^*\| \leq \|\{f^*(f_k)\}\|_{\mathcal{Z}_d} \leq \mathrm{B}_0\|f^*\|, \ for\ each\ f^* \in \mathcal{X}^*;$$

(iii) T is a bounded linear operator such that

$$T(\{f^*(f_k)\}) = f^*, \quad \text{for all } f^* \in \mathcal{X}^* .$$

The positive constants A_0, B_0 are called *retro frame bounds* of $(\{f_k\}, T)$ and the operator $T : \mathcal{Z}_d \to \mathcal{X}^*$ is called a *retro pre-frame operator* (or reconstruction operator) associated with the retro Banach frame and the operator $R : f^* \to \{f^*(f_k)\},\ f^* \in \mathcal{X}^*$, is called the *analysis operator* associated with the frame. A retro Banach frame $(\{f_k\}, T)$ is said to be an *exact* if there exists no reconstruction operator T_0 such that $(\{f_k\}_{k\neq m}, T_0)$ is a retro Banach frame for $\mathcal{X}^*$, where $m \in \mathbb{N}$ be arbitrary.

Lemma 1.1. ([10]). *Let $\mathcal{X}$ be a Banach space and $\{f_n\} \subset \mathcal{X}^*$ be a sequence such that $\{x \in \mathcal{X} : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$. Then $\mathcal{X}$ is linearly isometric to the Banach space $X = \{\{f_n(x)\} : x \in \mathcal{X}\}$, where the norm is given by $\|\{f_n(x)\}\|_X = \|x\|_{\mathcal{X}}$, $x \in \mathcal{X}$.*

Lemma 1.2. ([10]). *Let $(\{f_n\}, T)$ be a retro Banach frame for $\mathcal{X}^*$. Then $(\{f_n\}, T)$ is an exact if and only if $f_n \notin [f_i]_{i\neq n}$, for all $n \in \mathbb{N}$.*

1.1 Goal of the paper

If a signal is transmitted in the space, then there is an error which is hidden in the coefficients given by the reconstruction system. One of such reconstruction system is a frame which can reconstruction the underlying space by mean of an infinite series (or by an operator). Consider a discrete signal space $\mathbb{H} = L^2(\Omega)$, where $\Omega = \mathbb{N}$ with counting measure. Let $\{\Phi_n\} \subset \mathbb{H}$ be a Hilbert frame for $\mathbb{H}$. Let Φ be a non zero signal in $\mathbb{H}$. Then, Φ can be recovered by reconstruction the formula in (1.1). If we transmit the signal Φ into space, then it is in the form of frame coefficients $\{\langle \Phi, \Phi_n\rangle\}_n$. An error is always expected, let it be $\mathcal{E} = \{\xi_n\}$. Therefore, the signal received by the receiver is $\{\langle \Phi, \Phi_n\rangle + \xi_n\}_n$. So, the original signal, in general, can not be written a linear combination of $\{\Phi_n\}$ over $\{\langle \Phi, \Phi_n\rangle + \xi_n\}_n$. On the other hand a retro Banach frame reconstruct the signal by a retro pre-frame operator.

Let $\{e_n\}$ be an orthonormal basis for the discrete signal space $\mathbb{H}$. Choose $\Phi_n = e_n$, for all $n \in \mathbb{N}$. Then, $\{\Phi_n\}$ is an exact Hilbert frame for $\mathbb{H}$. Let S be the frame operator for $\{\Phi_n\}$. Fix $0 \neq \Phi \in \mathbb{H}$. Consider the error $\mathcal{E} = \{\langle \Phi, -\Phi_1 + \Phi_{n+1}\rangle\}_n$. This makes sense, because errors are not constant! Then, not all elements of $\mathbb{H}$ can be reconstructed by $\{\Phi_n\}$ over $\{\langle \Phi, \Phi_n\rangle + \xi_n\}_n$. In particular, it is easy to check that Φ_1 can not be written as a linear combination of $\{\Phi_n\}$ over $\{\langle \Phi, \Phi_n\rangle + \xi_n\}_n$. The family of vectors corresponding to the information given in $\{\langle \Phi, \Phi_n\rangle + \xi_n\}_n$ (of the disturbed signal) is $\{\Psi_n \equiv \Phi_n + (-\Phi_1 + \Phi_{n+1})\}$. Note that there exists no reconstruction operator Θ_0 such that $\mathcal{F}_0 \equiv (\{\Psi_n\}, \Theta_0)$ is a retro Banach frame for $\mathbb{H}^*$. Indeed, if a_0 and b_0 be retro frame bounds for $\mathcal{F}_0$, then

$$a_0\|\Phi^*\|_{\mathbb{H}^*} \leq \|\{\Phi^*(\Psi_n)\}\|_{\mathcal{Z}_d} \leq b_0\|\Phi^*\|_{\mathbb{H}^*}, \text{ for each } \Phi^* \in \mathbb{H}^*. \tag{1.2}$$

If $\Phi_0^* \in \mathbb{H}^*$ is given by $\Phi_0^*(\{\beta_j\}) = \begin{cases} \beta_1, & j = 1 \\ \beta_{j+1} - \beta_j, & j > 1, j \in \mathbb{N} \end{cases}$, $\{\beta_j\} \in \mathbb{H}$, then Φ_0^* is a non zero functional in $\mathbb{H}^*$. Put $\Phi^* = \Phi_0^*$ in retro frame inequality (1.2), we obtain $\Phi_0^* = 0$. This is a contradiction.

By giving suitable weight to the vectors obtained from the disturbed frame for the signal space $\mathbb{H}$, we can make the reconstruction operator such that each vector (signal) of $\mathbb{H}$ can be reconstructed by the said reconstruction operator (see Example 2.2).

2 Weighted Retro Banach Frames

Definition 2.1. Let $\omega = \{\omega_n\}$ be a weight. A pair $\mathcal{F} \equiv (\{\omega_n\Phi_n\}, \Theta)$ ($\{\Phi_n\} \subset \mathbb{H}$, $\Theta : \mathcal{X}_d \to \mathbb{H}^*$) is called a *weighted retro Banach frame* (or *ω-retro Banach frame*) for $\mathbb{H}^*$ with respect to $\mathcal{X}_d$ if:

(i) $\{\Phi^*(\omega_n\Phi_n)\} \in \mathcal{X}_d$, for each $\Phi^* \in \mathbb{H}^*$

(ii) there exist positive constants a_0 and b_0 ($0 < \mathrm{a}_0 \leq \mathrm{b}_0 < \infty$) such that

$$\mathrm{a}_0\|\Phi^*\|_{\mathbb{H}^*} \leq \|\{\Phi^*(\omega_n\Phi_n)\}\|_{\mathcal{X}_d} \leq \mathrm{b}_0\|\Phi^*\|_{\mathbb{H}^*}, \quad \text{for all } \Phi^* \in \mathbb{H}^* \tag{2.1}$$

(iii) Θ is a bounded linear operator such that

$$\Theta(\{\Phi^*(\omega_n\Phi_n)\}) = \Phi^*, \quad \text{for all } \Phi^* \in \mathbb{H}^*.$$

Regarding existence of weighted retro Banach frames we have following examples.

Example 2.1. *Let $\mathcal{X} = \mathbb{H}$ and let $\Phi_n = e_n, n \in \mathbb{N}$, where $\{e_n\}$ is the canonical basis for $\mathbb{H}$. Put $\omega_n = \frac{1}{n^2}, n \in \mathbb{N}$. Then, $\{\omega_n\}$ is a weight.*

Put $\mathcal{A}_d = \{\{\Phi^*(\omega_n\Phi_n)\} : \Phi^* \in \mathbb{H}^*\}$. Then, $\mathcal{A}_d$ is a Banach space with norm given by

$$\|\{\Phi^*(\omega_n\Phi_n)\}\|_{\mathcal{A}_d} = \|\Phi^*\|_{\mathbb{H}^*}.$$

Therefore, $\Theta : \{\Phi^*(\omega_n\Phi_n)\} \to \Phi^*$ is a bounded linear operator from $\mathcal{A}_d$ onto $\mathbb{H}^*$ such that $\mathcal{F} \equiv (\{\omega_n\Phi_n\}, \Theta)$ is a weighted retro Banach frame for $\mathbb{H}^*$ with respect to $\mathcal{A}_d$.

2.1 Application

The following example gives an application of a weighted retro Banach frame for a discrete signal space.

Example 2.2. *Consider a retro Banach frame* $(\{\Phi_n\}, \Theta)$ *for a discrete signal space* $\mathbb{H}$ *given in subsection (1.1). Let us give suitable weight to* $\mathcal{F}_0 = (\{\Psi_n\}, \Theta_0)$*, say* $\omega = \{\omega_n\} = \{2, 1, 1, 1, 1, ...\}$.

Choose $\mathcal{Z}_{d_0} = \{\{\Phi^*(\omega_n\Phi_n + (-\Phi_1 + \Phi_{n+1}))\} : \Phi^* \in \mathbb{H}^*\}$*. Then by Lemma* 1.1*,* $\mathcal{Z}_{d_0}$ *is a Banach space with norm given by*

$$\|\{\Phi^*(\omega_n\Phi_n + (-\Phi_1 + \Phi_{n+1}))\}\|_{\mathcal{Z}_{d_0}} = \|\Phi^*\|_{\mathbb{H}^*}.$$

Define $\widehat{\Theta}_0 : \mathcal{Z}_{d_0} \to \mathbb{H}^*$ *by*

$$\widehat{\Theta}_0(\{\Phi^*(\omega_n\Phi_n + (-\Phi_1 + \Phi_{n+1}))\}) = \Phi^*, \Phi^* \in \mathbb{H}^*.$$

Then, $\widehat{\Theta}_0 \in \mathcal{B}(\mathcal{Z}_{d_0}, \mathbb{H}^*)$*. Hence* $(\{(\omega_n\Phi_n + (-\Phi_1 + \Phi_{n+1}))\}, \widehat{\Theta}_0)$ *is a weighted retro Banach frame for* $\mathbb{H}^*$*. Thus, by giving suitable weight to a given Hilbert frame and knowing the nature of vectors which appear in the error, we can find a reconstruction operator which can reconstruct each vector of the underlying signal space. This reflects the importance of weighted retro Banach frames.*

The following proposition provides necessary and sufficient conditions for an exactness of a weighted retro Banach frame.

Proposition 2.1. *Let* $\omega = \{\omega_n\}$ *be a weight and let* $\mathcal{F} \equiv (\{\omega_n\Phi_n\}, \Theta)$ *be a weighted retro Banach frame for* $\mathbb{H}^*$ *with respect to* $\mathcal{Z}_d$*. Then* $\mathcal{F}$ *is an exact if and only if* $\omega_n\Phi_n \notin [\omega_i\Phi_i]_{i\neq n}$ *for all* $n \in \mathbb{N}$.

Proof. Similar to the proof of Lemma 1.2. □

Now we give necessary condition under for a weighted retro Banach frame to be an exact.

Proposition 2.2. *If* $\mathcal{F} \equiv (\{\omega_n\Phi_n\}, \Theta)$ *is an exact weighted retro Banach frame for* $\mathbb{H}^*$ *with respect to* $\mathcal{Z}_d$*, then*

$$\lim_{n\to\infty}\sum_{i=1}^{m_n}\alpha_i^{(n)}\omega_i\Phi_i = 0 \Rightarrow \lim_{n\to\infty}\alpha_j^{(n)} = 0 \ (j \in \mathbb{N}).$$

Proof. Since $\mathcal{F}$ is an exact. Therefore, by Hahn-Banach theorem there exists $\{\Psi_n^*\} \subset \mathbb{H}^*$ such that $\Psi_n^*(\omega_m\Phi_m) = \delta_{n,m}$, for all $n, m \in \mathbb{N}$. Also by using lower frame inequality for $\mathcal{F}$, we have $[\omega_n\Phi_n] = \mathbb{H}^*$. So, $\lim_{n\to\infty}\sum_{i=1}^{m_n}\alpha_i^{(n)}\omega_i\Phi_i \in \mathbb{H}^*$.
We compute

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{m_n}\alpha_i^{(n)}\omega_i\Phi_i = 0\\
&\Rightarrow \Psi_j^*(\lim_{n\to\infty}\sum_{i=1}^{m_n}\alpha_i^{(n)}\omega_i\Phi_i) = 0, \text{for each } j \in \mathbb{N}\\
&\Rightarrow \lim_{n\to\infty}\alpha_j^{(n)} = \Psi_j^*(\lim_{n\to\infty}\sum_{i=1}^{m_n}\alpha_i^{(n)}\omega_i\Phi_i) = 0, \text{for each } j \in \mathbb{N}.
\end{aligned}$$

This complete the proof. □

Remark 2.1. Let $\mathcal{F} \equiv (\{\omega_n \Phi_n\}, \Theta)$ and $\{\Psi_n^*\} \subset \mathbb{H}^*$ be same as in the Proposition 2.2. Then, in general, $\widetilde{[\Psi_n^*]} \neq \mathbb{H}^*$. If $\widetilde{[\Psi_n^*]} = \mathbb{H}^*$, then

$$\lim_{n\to\infty} \alpha_j^{(n)} = 0 \ (j \in \mathbb{N}) \Rightarrow \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} \omega_i \Phi_i = 0.$$

Indeed, for a fixed $j \in \mathbb{N}$, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \alpha_j^{(n)} \\ &= \Psi_j^*(\lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} \omega_i \Phi_i) \\ &= \Psi_j^*(\Phi_0), \text{where } \Phi_0 = \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} \omega_i \Phi_i. \end{aligned}$$

Thus, $\Psi_j^*(\Phi_0) = 0,$ for all $j \in \mathbb{N}$. Therefore by using the fact that $\widetilde{[\Psi_n^*]} = \mathbb{H}^*$, we obtain $\Phi_0 = 0$.

The following theorem gives necessary and sufficient condition for a sequence $\{\Phi_n\}$ in $\mathbb{H}$ to be a weighted retro Banach frame.

Theorem 2.3. *Let $\{\omega_n\}$ be a weight and let $\{\Phi_n\} \subset \mathbb{H}$. A system $\mathcal{F} \equiv (\{\omega_n \Phi_n\}, \Theta)$ is a weighted retro Banach frame for $\mathbb{H}^*$ if and only if dist$(\Phi, L_n) \to 0$ as $n \to \infty$, for all $\Phi \in \mathbb{H}$, where $L_n = [\omega_1\Phi_1, \omega_2\Phi_2, ..., \omega_n\Phi_n]$, for all $n \in \mathbb{N}$.*

Proof. Suppose first that $\mathcal{F} \equiv (\{\omega_n \Phi_n\}, \Theta)$ is a weighted retro Banach frame. Let A_0 and B_0 be a choice of bounds for $\mathcal{F}$.
Then

$$A_0\|\Phi^*\|_{\mathbb{H}^*} \leq \|\{\Phi^*(\omega_n\Phi_n)\}\|_{(\mathbb{H}^*)_d} \leq B_0\|\Phi^*\|_{\mathbb{H}^*}, \quad \text{for all } \Phi^* \in \mathbb{H}^*. \tag{2.2}$$

Suppose that the condition dist$(\Phi, L_n) \to 0$ as $n \to \infty$, for all $\Phi \in \mathbb{H}$, is not satisfied. Then, there exists a non zero functional $\Phi_0 \in \mathbb{H}$ such that

$$\lim_{n\to\infty} \text{dist}(\Phi_0, L_n) \neq 0.$$

Note that dist$(\Phi_0, L_n) \geq$ dist(Φ_0, L_{n+1}) for all $n \in \mathbb{N}$. This is because $\{L_n\}$ is a nested system of subspaces and by definition of distance of a point from a set. Now $\{$dist$(\Phi_0, L_n)\}$ is coercive monotone decreasing sequence. So, $\{$dist$(\Phi_0, L_n)\}$ is convergent and its limit is a positive real number, since otherwise dist$(\Phi_0, L_n) \to 0$ as $n \to \infty$ (which is not possible). Let $\lim_{n\to\infty}$ dist$(\Phi_0, L_n) = \xi > 0$.

Choose $D = \bigcup_n L_n$. Then, by using the fact that dist$(\Phi_0, D) = \inf\{$dist$(\Phi_0, L_n)\}$, we obtain

$$\text{dist}(\Phi_0, D) \geq \xi > 0. \tag{2.3}$$

Now we show that $\Phi_0 \notin \overline{D}$. Let if possible, $\Phi_0 \in \overline{D}$. Then, we can find a sequence $\{\zeta_n\} \subset D$ such that dist$(\zeta_n, \Phi_0) \to 0$ as $n \to \infty$.

By using (2.3), we have dist$(\Phi_0, D) \geq \xi$. Therefore, dist$(\zeta_n, \Phi_0) \geq \xi > 0$. This is a contradiction to the fact that dist$(\zeta_n, \Phi_0) \to 0$ as $n \to \infty$. Hence $\Phi_0 \notin \overline{D}$. Thus, by using Hahn-Banach theorem, there exists a non zero functional $\Phi_0^* \in \mathbb{H}^*$ such that

$$\Phi_0^*(\omega_n\Phi_n) = 0, \text{ for all } n \in \mathbb{N}.$$

Therefore, by using retro frame inequality (2.2), we have $\Phi_0^* = 0$, a contradiction. Hence dist$(\Phi, L_n) \to 0$ as $n \to \infty$, for all $\Phi \in \mathbb{H}$.

To prove the converse part, assume that there exists a system $\{\Phi_n\} \subset \mathbb{H}$ is such that

$$\text{dist}(\Phi, L_n) \to 0 \text{ as } n \to \infty, \text{ for all } \Phi \in \mathbb{H}.$$

Then, in particular, for each $\epsilon > 0$ and for each $\Phi \in \mathbb{H}$, we can find a $\omega_j \Phi_j$ from some L_k such that

$$\|\Phi - \omega_j \Phi_j\| < \epsilon,$$

which gives $\{\omega_j \Phi_j\}$ is complete in $\mathbb{H}$. Therefore, by using Lemma 1.1, $\mathcal{Z} = \{\{\Phi^*(\omega_n \Phi_n)\} : \Phi^* \in \mathbb{H}^*\}$ is a Banach space of sequences of scalars with norm given by

$$\|\{\Phi^*(\omega_n \Phi_n)\}\|_{\mathcal{Z}} = \|\Phi^*\|_{\mathbb{H}^*}, \ \Phi^* \in \mathbb{H}^*.$$

Define $\Theta_0 : \mathcal{Z} \to \mathbb{H}^*$ by

$$\Theta_0(\{\Phi^*(\omega_n \Phi_n)\}) = \Phi^*, \ \Phi^* \in \mathbb{H}^*.$$

Then, Θ_0 is a bounded linear operator such that $(\{\omega_n \Phi_n\}, \Theta_0)$ is a weighted retro Banach frame for $\mathbb{H}^*$ with respect to $\mathcal{Z}$. □

The following theorem gives necessary and sufficient conditions for the existence of a weighted retro Banach frame for discrete signal space with respect to a given associated Banach space of scalar valued sequences $\mathcal{Z}_d$.

Theorem 2.4. *A system $\mathcal{F} \equiv (\{\omega_n \Phi_n\}, \Theta)$ is a weighted retro Banach frame for $\mathbb{H}^*$ with respect to $\mathcal{Z}_d$ which is generated by $\{\Phi^*(\omega_n \Phi_n)\}$ if and and only if $\mathbb{H}^*$ is isomorphic to a closed subspace of $\mathcal{Z}_d$.*

Proof. Assume first that $\mathcal{F} \equiv (\{\omega_n \Phi_n\}, \Theta)$ is a weighted retro Banach frame for $\mathbb{H}^*$ with respect to $\mathcal{Z}_d$. Then, there exists positive constants A, B such that

$$A\|\Phi^*\| \leq \|\{\Phi^*(\omega_n \Phi_n\}\|_{\mathcal{Z}_d} \leq B\|\Phi^*\|, \text{ for all } \Phi^* \in \mathbb{H}^*. \tag{2.4}$$

By using lower frame inequality in (2.4), the analysis operator T of $\{\omega_n \Phi_n\}$ is coercive. Thus T is injective and has closed range. From the Inverse Mapping Theorem, $\mathbb{H}^*$ is isomorphic to the range $T(\mathbb{H}^*)$, which is a subspace of $\mathcal{Z}_d$. For the reverse part, assume that M is a closed subspace of $\mathcal{Z}_d$ and U is an isomorphic from $\mathbb{H}^*$ onto M. Let $\{P_i\}$ be the sequence of coordinate operators on $\mathcal{Z}_d$, then $P_i(\{y_j\})$=y_i for all $i \in \mathbb{N}$.
Choose $\Phi^*(\omega_n \Phi_n)$=$P_n U(\Phi^*), n \in \mathbb{N}$. Then, for all $\Phi^* \in \mathbb{H}^*$ we have

$$\|\Phi^*\| = \|U^{-1}U(\Phi^*)\| \leq \|U^{-1}\|\|U(\Phi^*)\|.$$

Therefore

$$\frac{\|\Phi^*\|}{\|U^{-1}\|} \leq \|\{\Phi^*(\omega_n \Phi_n)\}\|_{\mathcal{Z}_d} = \|\{P_n U(\Phi^*)\}\|_{\mathcal{Z}_d} = \|U(\Phi^*)\| \leq \|U\|\|\Phi^*\|, \ \forall \ \Phi^* \in \mathbb{H}^*.$$

Hence $\{\omega_n \Phi_n\}$ is a weighted retro Banach frame for $\mathbb{H}^*$ with respect to $\mathcal{Z}_d$. □

2.2 Construction of weighted retro Banach frames for $\mathbb{H}^*$ from bounded linear operators on $\mathcal{Z}_d$

Let $\mathcal{F} \equiv (\{\omega_n \Phi_n\}, \Theta)$ be a weighted retro Banach frame for $\mathbb{H}^*$ with respect to $\mathcal{Z}_d$ and let $\{\Psi_k\} \subset \mathbb{H}$. Let $\widehat{\Theta}$ be a bounded linear operator on $\mathcal{Z}_d$ such that $\widehat{\Theta}(\{\Phi^*(\omega_k \Phi_k)\}) = \{\Phi^*(\omega_k \Psi_k)\}, \Phi^* \in \mathbb{H}^*$. Then, in general, there exists no reconstruction operator Θ_0 such that $(\{\omega_k \Psi_k\}, \Theta_0)$ is a weighted retro Banach frame for $\mathbb{H}^*$ with respect to $\mathcal{Z}_d$. The following theorem provides necessary and sufficient condition for the construction of a weighted retro Banach frame for $\mathbb{H}^*$ with respect to $\mathcal{Z}_d$.

Theorem 2.5. *Let $(\{\omega_k\Phi_k\},\Theta)$ be a weighted retro Banach frame for $\mathbb{H}^*$ with respect to $\mathcal{Z}_d$ and let $\{\Psi_k\}$ be a sequence in $\mathbb{H}$ such that $\{\Phi^*(\omega_k\Psi_k)\}\in\mathcal{Z}_d,\ \Phi^*\in\mathbb{H}^*$. Assume that $\widehat{\Theta}$ is a bounded linear operator on $\mathcal{Z}_d$ such that $\widehat{\Theta}(\{\Phi^*(\omega_k\Phi_k)\})=\{\Phi^*(\omega_k\Psi_k)\},\ \Phi^*\in\mathbb{H}^*$, where $\{\Psi_k\}\subset\mathbb{H}$. Then, there exists a reconstruction operator Θ_0 such that $(\{\omega_k\Psi_k\},\Theta_0)$ is a weighted retro Banach frame for $\mathbb{H}^*$ with respect to $\mathcal{Z}_d$ if and only if*

$$\|\widehat{\Theta}(\{\Phi^*(\omega_k\Phi_k)\})\|_{\mathcal{Z}_d}\geq\gamma\,\|Q(\{\Phi^*(\omega_k\Psi_k)\})\|_{\mathcal{Z}_d},\ \Phi^*\in\mathbb{H}^*,$$

where γ is a positive constant and Q is a bounded linear operator on Z_d such that $Q(\{\Phi^*(\omega_k\Psi_k)\})=\{\Phi^*(\omega_k\Phi_k)\},\Phi^*\in\mathbb{H}^*$.

Proof. Assume first that $\mathcal{G}\equiv(\{\omega_k\Psi_k\},\Theta_0)$ is a weighted retro Banach frame for $\mathcal{Z}_d$ with bounds a_0,b_0. Let S and T be the pre-frame operator and analysis operator associated with $\{\omega_k\Phi_k\}$, respectively. Choose $Q=TS$. Then, Q is a bounded linear operator on $\mathcal{Z}_d$ is such that $Q(\{\Phi^*(\omega_k\Psi_k)\})=\{\Phi^*(\omega_k\Phi_k)\},\ \Phi^*\in\mathbb{H}^*$. Let S_0 be the pre-frame operator associated with $\mathcal{G}$. Choose $\gamma=\frac{\|S_0\|^{-1}}{\|T\|}$. Then, for all $\Phi^*\in\mathbb{H}^*$ we have

$$\begin{aligned}\|\widehat{\Theta}(\{\Phi^*(\omega_k\Phi_k)\})\|_{\mathcal{Z}_d}&=\|\{\Phi^*(\omega_k\Psi_k)\}\|_{\mathcal{Z}_d}\\&\geq\|S_0\|^{-1}\|\Phi^*\|\\&\geq\gamma\|Q(\{\Phi^*(\omega_k\Psi_k)\})\|_{\mathcal{Z}_d}.\end{aligned}$$

For the reverse part, we compute

$$\begin{aligned}\gamma\,\|S_0\|^{-1}\|\Phi^*\|&\leq\gamma\,\|\{\Phi^*(\omega_k\Phi_k)\}\|_{\mathcal{Z}_d}\\&=\gamma\|Q(\{\Phi^*(\omega_k\Psi_k)\})\|_{\mathcal{Z}_d}\\&\leq\|\widehat{\Theta}(\{\Phi^*(\omega_k\Phi_k)\})\|_{\mathcal{Z}_d}\ (=\|\{\Phi^*(\omega_k\Psi_k)\}\|_{\mathcal{Z}_d})\\&\leq\|\widehat{\Theta}\|\,\|T\|\,\|\Phi^*\|,\text{ for all }\Phi^*\in\mathbb{H}^*.\end{aligned}$$

Put $\Theta_0=\Theta Q$. Then, $\Theta_0:\mathcal{Z}\to\mathbb{H}^*$ is a bounded linear operator such that $\Theta_0(\{\Phi^*(\omega_k\Psi_k)\})=\Phi^*,\forall\Phi^*\in\mathbb{H}^*$. Hence $(\{\omega_k\Psi_k\},\Theta_0)$ is a weighted retro Banach frame for $\mathbb{H}^*$ with respect to $\mathcal{Z}_d$. The theorem is proved. □

The following theorem gives the better Bessel bound for the sum of two weighted retro Banach frames.

Theorem 2.6. *Let $\mathcal{F}\equiv(\{\omega_k\Phi_k\},\Theta)$ and $\mathcal{G}\equiv(\{\omega_k\Psi_k\},\Theta_0)$ be weighted retro Banach frames for $\mathbb{H}^*$ with respect to $\mathcal{Z}_d$ and let T be a bounded linear invertible operator on $\mathcal{Z}_d$ such that $T(\{\Phi^*(\omega_k\Phi_k)\})=\{\Phi^*(\omega_k\Psi_k)\},\ \Phi^*\in\mathbb{H}^*$. Then, $\{\omega_k\Phi_k+\omega_k\Psi_k\}$ is a weighted retro Bessel sequence with bound*

$$\alpha=\min\{\|\widehat{\Theta}\|\|I+T\|,\|\widehat{\Theta}_0\|\|I+T^{-1}\|\},$$

where $\widehat{\Theta},\ \widehat{\Theta}_0$ are the analysis operators associated with $\mathcal{F}$ and $\mathcal{G}$, respectively and I is the identity operator on $\mathcal{Z}_d$.

Proof. For all $\Phi^*\in\mathbb{H}^*$, we have

$$\begin{aligned}\|\{\Phi^*(\omega_k\Phi_k+\omega_k\Psi_k)\}\|_{\mathcal{Z}_d}&=\|(I+T)(\{\Phi^*(\omega_k\Phi_k)\})\|_{\mathcal{Z}_d}\\&\leq\|I+T\|\|\widehat{\Theta}\|\|\Phi^*\|.\end{aligned}$$

Similarly, we can show that

$$\|\{\Phi^*(\omega_k\Phi_k+\omega_k\Psi_k)\}\|_{\mathcal{Z}_d}\leq\|I+T^{-1}\|\|\widehat{\Theta}_0\|\|\Phi^*\|,\text{ for all }\Phi^*\in\mathbb{H}^*.$$

Hence $\{\omega_k\Phi_k+\omega_k\Psi_k\}$ is a weighed retro Bessel sequence with required bound. □

Remark 2.2. For the weighted retro Bessel sequence $\{\omega_k\Phi_k + \omega_k\Psi_k\}$ in Theorem 2.6, in general, there exists no reconstruction operator U such that $(\{\omega_k\Phi_k + \omega_k\Psi_k\}, U)$ is a weighted retro Banach frame for $\mathbb{H}^*$. If the analysis operator associated with weighted Bessel sequence is coercive, then the respective weighted retro Bessel sequence constitute a weighted retro Banach frame for the discrete signal space. This is summarized in the following lemma.

Lemma 2.7. *Let $\{\Upsilon_k\} \subset \mathbb{H}$ be a weighted retro Bessel sequence for $\mathbb{H}^*$, then there exists a reconstruction operator U such that $(\{\Upsilon_k\}, U)$ is a weighted retro Banach frame for $\mathbb{H}^*$ if and only if its analysis operator is coercive.*

Proof. Proof is similar to proof in Theorem 2.5.

2.3 Relation between weighted retro Bessel bounds and weighted retro Banach frame bounds:

In signal processing there is an overlapping between the signal under consideration and other signals of high frequency (or energy). Here we consider sum or difference (algebraic overlapping) of a weighted retro Banach frame with weighted retro Bessel sequence in discrete signal space. It is interesting to know the relation between weighted retro Bessel bounds and weighted retro Banach frame bounds such that after overlapping (algebraic operations) the resultant system constitute a weighted retro Banach frame for the discrete signal space . In this direction, the following proposition provides a result in Banach space setting.

Proposition 2.3. *Let $\mathcal{F} \equiv (\{\omega_k\Phi_k\}, \Theta)$ be a weighted retro Banach frames for $\mathbb{H}^*$ with respect to $\mathcal{Z}_d$ and with bounds A, B and let $\{\omega_k\Psi_k\}$ be a weighted retro Bessel sequence for $\mathbb{H}^*$ with respect to $\mathcal{Z}_d$ with bound M. Then, $\{\omega_k(\Phi_k \pm \Psi_k)\}$ is a weighted retro Banach frames for $\mathbb{H}^*$, provided $M < A$.*

Proof. Let T and R be the analysis operators associated with $\{\omega_k\Phi_k\}$ and $\{\omega_k\Psi_k\}$, respectively. We compute

$$\begin{aligned}\|\{\Phi^*(\omega_k\Phi_k \pm \omega_k\Psi_k)\}\|_{\mathcal{Z}_d} &= \|(T \pm R)\Phi^*\|_{\mathcal{Z}_d} \\ &= \|T(\Phi^*) \pm R(\Phi^*)\|_{\mathcal{Z}_d} \\ &\geq \|T(\Phi^*)\|_{\mathcal{Z}_d} - \|R(\Phi^*)\|_{\mathcal{Z}_d} \\ &\geq (A - M)\|\Phi^*\|, \text{ for all } \Phi^* \in \mathbb{H}^*.\end{aligned}$$

Similarly, we have

$$\|\{\Phi^*(\omega_k\Phi_k \pm \omega_k\Psi_k)\}\|_{\mathcal{Z}_d} \leq (B + M)\|\Phi^*\|, \text{ for all } \Phi^* \in \mathbb{H}^*.$$

Choose $\mathcal{Y}_d = \{\{\Phi^*(\omega_k\Phi_k \pm \omega_k\Psi_k)\} : \Phi^* \in \mathbb{H}^*\}$. Then, $\mathcal{Y}_d$ is a Banach space with the norm given by

$$\|\{\Phi^*(\omega_k\Phi_k \pm \omega_k\Psi_k)\}\|_{\mathcal{Y}_d} = \|\Phi^*\|_{\mathbb{H}^*}, \ \Phi^* \in \mathbb{H}^*.$$

Define $\Theta_0 : \mathcal{Y}_d \to \mathbb{H}^*$ by $\Theta_0(\{\Phi^*(\omega_k\Phi_k \pm \omega_k\Psi_k)\}) = \Phi^*$. Then, Θ_0 is a bounded linear operator such that $(\{\omega_k(\Phi_k \pm \Psi_k)\}, \Theta_0)$ is a weighted retro Banach frames for $\mathbb{H}^*$. The proposition is proved. □

Let $\mathcal{F} \equiv (\{\omega_k\Phi_k\}, \Theta)$ be a weighted retro Banach frames for $\mathbb{H}^*$ with respect to $\mathcal{Z}_d$ and let $\{\Psi_k\} \subset \mathbb{H}$. The following proposition give an estimate of a weighted retro Bessel bound for $\{\omega_k\Phi_k + \omega_k\Psi_k\}$ such that $(\{\Psi_k\}, \Theta_0)$ is a weighted retro Banach frames for $\mathbb{H}^*$.

Proposition 2.4. *Assume that $\mathcal{F} \equiv (\{\omega_k\Phi_k\}, \Theta)$ is a weighted retro Banach frame for $\mathbb{H}^*$ with respect to $\mathcal{Z}_d$ and $\{\Psi_k\} \subset \mathbb{H}$. If $\{\omega_k\Phi_k + \omega_k\Psi_k\}$ is a weighted retro Bessel sequence for $\mathbb{H}^*$ with Bessel bound $\delta < \|\Theta\|^{-1}$. Then, there exists a reconstruction operator Θ_0 such that $(\{\omega_k\Psi_k\}, \Theta_0)$ is a weighted retro Banach frames for $\mathbb{H}^*$.*

Proof. Similar to proof of Proposition 2.3.

□

2.4 Perturbation of weighted retro Banach frames

Perturbation theory is a very important tool in various areas of applied mathematics [1, 2, 11]. In frame theory, it began with the fundamental perturbation result of Paley and Wiener [12]. The basic of Paley and Wiener was that a system that is sufficient close to an orthonormal system (basis) in a Hilbert space is also form an orthonormal system (basis). Since then, a number of variations and generalization of this perturbation to the setting of Banach space [1,11] and then to perturbation of the atomic decompositions, frames (Hilbert)and Banach frames, the reconstruction property in Banach spaces [13, 14, 15, 16, 17, 18, 19, 20]. The following theorem gives a Paley-Wiener type perturbation (in Banach space setting) for weighted retro Banach frames.

Theorem 2.8. *Let $\mathcal{F} \equiv (\{\omega_k \Phi_k\}, \Theta)$ be a weighted retro Banach frames for $\mathbb{H}^*$ with respect to $\mathcal{Z}_d$ and with bounds A, B and let $\{\Psi_k\} \subset \mathbb{H}$.*

Assume that $\lambda, \mu, \nu \geq 0$ are non-negative real number such that $\max\left(\lambda + \frac{\nu}{A}, \mu\right) < 1$ and

$$\|(T - R)(\Phi^*)\|_{\mathcal{Z}_d} \leq \lambda \|T(\Phi^*)\|_{\mathcal{Z}_d} + \mu \|R(\Phi^*)\|_{\mathcal{Z}_d} + \nu \|\Phi^*\|, \text{ for all } \Phi^* \in \mathbb{H}^*. \quad (2.5)$$

where T and R the analysis operators associated with $\{\omega_k \Phi_k\}$ and $\{\omega_k \Psi_k\}$, respectively. Then, there exists a reconstruction operator Θ_0 such that $(\{\omega_k \Psi_k\}, \Theta_0)$ is a weighted retro Banach frames for $\mathbb{H}^*$ with bounds $\left(\frac{(1-\lambda)A-\nu}{1+\mu}\right)$ and $\left(\frac{(1+\lambda)B+\nu}{1-\mu}\right)$ with respect to a Banach space generated by $\{\{R(\Phi^*)\} : \Phi^* \in \mathbb{H}^*\}$.

Proof. For any $\Phi^* \in \mathbb{H}^*$, we have

$$A\|\Phi^*\| \leq \|\{\Phi^*(\omega_k \Phi_k)\}\|_{\mathcal{Z}_d} \leq B\|\Phi^*\|, \text{ for all } \Phi^* \in \mathbb{H}^*.$$

Since

$$\|(T - R)(\Phi^*)\|_{\mathcal{Z}_d} \geq \|R(\Phi^*)\|_{\mathcal{Z}_d} - \|T(\Phi^*)\|_{\mathcal{Z}_d}. \quad (2.6)$$

By using (2.5) and (2.6), we have

$$\begin{aligned} \|R(\Phi^*)\|_{\mathcal{Z}_d} &\leq \frac{1+\lambda}{1-\mu}\|T(\Phi^*)\|_{\mathcal{Z}_d} + \frac{\nu}{1-\mu}\|\Phi^*\| \\ &\leq \left(\frac{(1+\lambda)B+\nu}{1-\mu}\right)\|\Phi^*\|, \text{ for all } \Phi^* \in \mathbb{H}^*. \end{aligned} \quad (2.7)$$

Now

$$\|(T - R)(\Phi^*)\|_{\mathcal{Z}_d} \geq \|T(\Phi^*)\|_{\mathcal{Z}_d} - \|R(\Phi^*)\|_{\mathcal{Z}_d}, \Phi^* \in \mathbb{H}^*. \quad (2.8)$$

By using (2.5) and (2.8), we have

$$\begin{aligned} \|R(\Phi^*)\|_{\mathcal{Z}_d} &\geq \frac{1-\lambda}{1+\mu}\|T(\Phi^*)\|_{\mathcal{Z}_d} - \frac{\nu}{1+\mu}\|\Phi^*\| \\ &\geq \left(\frac{(1-\lambda)A-\nu}{1+\mu}\right)\|\Phi^*\|, \text{ for all } \Phi^* \in \mathbb{H}^*. \end{aligned} \quad (2.9)$$

Choose $\mathcal{Q}_d = \{\{R(\Phi^*)\} : \Phi^* \in \mathbb{H}^*\}$. Then, by using (2.7) and (2.9) $\mathcal{Q}_d$ is a Banach space with the norm given by

$$\|\{R(\Phi^*)\}\|_{\mathcal{Q}_d} = \|\Phi^*\|_{\mathbb{H}^*}, \ \Phi^* \in \mathbb{H}^*.$$

Define $\Theta_0 : \mathcal{Q}_d \to \mathbb{H}^*$ by $\Theta_0(\{R(\Phi^*)\}) = \Phi^*$. Then, Θ_0 is a bounded linear operator such that $(\{\omega_k \Psi_k\}, \Theta_0)$ is a weighted retro Banach frames for $\mathbb{H}^*$ with respect to $\mathcal{Q}_d$ and with desired frame bounds. □

Conclusion

Define weighted retro Banach frame for discrete signal space. We give a fruitful application of a weighted retro Banach frame for discrete signal space through an example. Obtain necessary and sufficient for a weighted retro Banach frame to be an exact. Also find out necessary and sufficient condition for a sequence in discrete signal space to be a weighted retro Banach frame. Construct weighted retro Banach frames from bounded linear operators. Relationship between weighted retro Bessel bounds and weighted retro Banach frame bounds is discussed. Lastly, we obtain a Paley-Wiener type perturbation for a weighted retro Banach frame in Banach space

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Competing Interests

The authors declare that no competing interests exist.

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