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Semi-analytical Approximation for Solving High-order Sturm-Liouville Problems

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Abstract

In this paper, an algorithm for solving high-order non-singular Sturm-Liouville eigenvalue problems is proposed. A modified form of Adomian decomposition method is implemented to provide a semi-analytical solution in the form of a rapidly convergent series. Convergent analysis and error estimate based on the Banach fixed-point is discussed. Five high-order Sturm-Liouville problems are solved numerically. Numerical results demonstrate reliability and efficiency of the proposed scheme.

Keywords: High-order Sturm-Liouville problems; Modified Adomian decomposition method; Banach fixed point theorem; Eigenvalues; Eigenfunctions.

2010 Mathematics Subject Classification: 34B24; 47H10; 34L10; 34L15; 34L16; 35C10

1 Introduction

In this study, we will propose an alternative semi-analytical approximation based on a new type of modified Adomian decomposition which is an application of the fixed point iteration method to solve

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non-singular high-order Sturm-liouville problems in the form

$$\begin{aligned}&(-1)^m(p_m(x)y^{(m)})^{(m)}+(-1)^{m-1}(p_{m-1}(x)y^{(m-1)})^{(m-1)}\\&\quad+\cdots+(p_2(x)y'')''-(p_1(x)y')'+p_0(x)y=\lambda w(x)y,\ \ a=0<x<b,\end{aligned} \tag{1.1}$$

subject to some $2m$ point specified conditions at the boundary $x\in\{a,b\}$ on

$$\begin{aligned}&u_k=y^{(k-1)},\ \ 1\le k\le m,\\&v_1=p_1y'-(p_2y'')'+(p_3y''')''+\cdots+(-1)^{m-1}(p_my^{(m)})^{(m-1)},\\&v_2=p_2y''-(p_3y''')'+(p_4y^{(4)})''+\cdots+(-1)^{m-2}(p_my^{(m)})^{(m-2)},\\&\vdots\\&v_k=p_ky^{(k)}-(p_{k+1}y^{(k+1)})'+(p_{k+2}y^{(k+2)})''+\cdots+(-1)^{m-k}(p_my^{(m)})^{(m-k)},\\&\vdots\\&v_m=p_my^{(m)}.\end{aligned} \tag{1.2}$$

In Eq. (1.1), we assume that all coefficient functions are real valued. The technical conditions for the problem to be non-singular are: the interval (a,b) is finite; the coefficient functions p_k $(0\le k\le m-1)$, $w(x)$ and $1/p_m(x)$ are in $L^1(a,b)$, $p_m(x)$ and weight function $w(x)$ are both positive. The eigenvalues $\lambda_k,\ \ k=1,\,2,\,3,\ldots$ can be ordered as an increasing sequence

$$\lambda_1\le\lambda_2\le\lambda_3\le\cdots,$$

where $\lim_{k\to\infty}\lambda_k=\infty$ and each eigenvalue has multiplicity at most m [1], [2]. The Sturm-Liouville boundary value problems play an important role in both theory and applications of ordinary differential equations. Many physical phenomena, both in classical mechanics and in quantum mechanics are described mathematically by second-order Sturm-Liouville problems [3], [4], [5]. However many important phenomena occurring in various fields of science are described mathematically by high-order Sturm-Liouville problems. For example, the free vibration analysis of beam structures [6], [7], [8] is governed by a fourth-order Sturm-Liouville problem, and it is known that when a layer of fluid is heated from below and is subject to the action of rotation, instability may set as overstability, this instability my be modelled by a eighth-order Sturm-Liouville boundary value problem with appropriate boundary conditions specified. It may be noted that, when instability sets as ordinary convection, the marginal state will be characterized by sixth-order Sturm-Liouville boundary value problem [9], [10], [11], [12]. Ten and twelfth-order Sturm-Liouville boundary value problems arise in the context when a uniform magnetic field is applied across the fluid in the same direction as gravity. When instability sets in as an ordinary convection, it is modelled by the tenth-order boundary value problems, when instability sets in as overstability, it is modelled by the twelfth-order boundary value problems [1], [2], [11], [12]. Let $L^2_w(a,b)$, be the space of functions $f(x)$ on (a,b) such that

$$\int_a^b f(x)|^2\,w(x)dx<\infty.$$

$L^2_w(a,b)$ is a Hilbert space with inner product

$$\langle f,g\rangle=\int_a^b f(x)\overline{g(x)}w(x)dx,$$

and norm $\|f\|^2=\langle f,f\rangle$. The standard Adomian decomposition method is applied for computing eigenvalues of Sturm-Liouville problems [13], [14], [15]. In the present work, based on basic idea of the Adomian decomposition method [16], [17], [18] and [19], we will improve a modified Adomian decomposition algorithm to solve high-order Sturm-Liouville problem (1.1) which is summarized in the following section. The paper is organized as follows: Modified Adomian decomposition method for solving high-order Sturm-Liouville problems is proposed in Section 2. Whill convergence of a new modification is discussed in Section 3. To illustrate the efficiency of proposed technique five numerical examples are discussed in Section 4. Section 5 concludes the paper.

2 Modified Adomian Decomposition Method (MADM)

Let us rewrite equation (1.1) in the following form

$$(-1)^m(p_m(x)y^{(m)})^{(m)} = F(y, y', \ldots, y^{(2m-2)}, \lambda) = (\lambda w(x) - p_0(x))y \\ -\{(-1)^{m-1}(p_{m-1}(x)y)^{(m-1)} + \cdots + (p_2(x)y'')''\}, \quad a < x < b, \tag{2.1}$$

which can be written in the operator form as

$$Ly(x) + Ry(x) = 0, \tag{2.2}$$

where $Ly(x) = (p_m(x)y^{(m)})^{(m)}$ and $Ry(x) = -F(y, y', \ldots, y^{(2m-1)}, \lambda)$, Ry is a differential operator satisfies Lipchitz condition for $y, \hat{y} \in L^2_w(a,b)$ and $C > 0$, we have, $\|Ry - R\hat{y}\| \leq C\|y - \hat{y}\|$. We define the differential operator

$$L = \frac{d^m}{dx^m}\left(p_m(x)\frac{d^m}{dx^m}\right), \tag{2.3}$$

then Eq. (2.1) can be rewritten as

$$Ly = F(y, y', \ldots, y^{(2m-2)}, \lambda), \tag{2.4}$$

The inverse operator L^{-1} is therefore considered a $2m$-fold integral operator defined by

$$L^{-1} = \underbrace{\int_0^x \int_0^{x_1} \cdots \int_0^{x_{m-1}}}_{m\ times} \frac{1}{p_m(x_m)} \underbrace{\int_0^{x_m} \int_0^{x_{m+1}} \cdots \int_0^{x_{2m-1}}}_{m\ times} dx_{2m} \ldots dx_1 \tag{2.5}$$

Operating with L^{-1} on (2.4), we get

$$y(x) = y_0(x) + L^{-1}F(y, y', \ldots, y^{2m-1}, \lambda). \tag{2.6}$$

The Adomian decomposition method expresses the solution $y(x)$ of (1.1) by the decomposition series

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{2.7}$$

The method defines $F(y, y', \ldots, y^{(2m-2)}, \lambda)$ by an infinite series of polynomials

$$F(y, y', \ldots, y^{(2m-2)}, \lambda) = \sum_{n=0}^{\infty} A_n(x, \lambda), \tag{2.8}$$

where $A_n(x, \lambda)$ are the so-called Adomian polynomials. Substituting (2.7) and (2.8) into (2.6), we have

$$\sum_{n=0}^{\infty} y_n(x) = y_0(x) + L^{-1}\left(\sum_{n=0}^{\infty} A_n(x, \lambda)\right). \tag{2.9}$$

The components of the series (2.7), $y_n(x),\ n \geq 0$, are obtained in the following recursive relation: by using all terms that arise from the boundary conditions at $x = a$ and from $Ly_0(x) = 0$, we determine $y_0(x)$, thus

$$y_0(x) = \sum_{i=0}^{2m-1} c_i \frac{x^i}{i!}, \tag{2.10}$$

where $c_i,\ i = 0, \ldots, 2m-1$ are some constants. Now by using Eq. (2.10), we can determined the remaining components by the following relation

$$y_{n+1} = \sum_{i=0}^{2m-1} c_i \frac{x^i}{i!} + L^{-1}(A_n(x, \lambda)),\ \ n \geq 0, \tag{2.11}$$

for determination of the components $y_n(x)$ of $y(x)$. In Eq. (2.11) $A_n(x, \lambda)$, are the Aomian polynomial defined as

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\mu^n}\left[R\left(\sum_{i=0}^{\infty}\mu^i y_i, \ldots, \sum_{i=0}^{\infty}\mu^i y_i^{(2m-2)}, \lambda\right)\right]\right]_{\mu=0}. \tag{2.12}$$

Fixed points of (2.11) under the suitable choice of the initial approximation $y_0(x)$ that is given by (2.10) are in fact solutions of problem (2.1). Note that exactly m conditions are specified initially at $x = a$, (these m conditions arise in different forms based on nature of the problem such as order of the highest derivative appearing in each condition must be less than $2m$). Now, if these m conditions at $x = a$ have the following form

$$y_n(a, \lambda) = y_n'(a, \lambda) = \cdots = y_n^{(m-1)}(a, \lambda) = 0,$$

then the approximate solution will be

$$y_n(x, \lambda) = \sum_{i=m}^{2m-1} c_i f_{n_i}(x, \lambda), \quad n > 0. \tag{2.13}$$

By using other conditions at endpoint b, for example $y_n(b, \lambda) = y_n'(b, \lambda) = \cdots = y_n^{(m-1)}(b, \lambda) = 0$, we get the following system

$$\begin{aligned} &\sum_{i=m}^{2m-1} c_i f_{n_i}(b, \lambda) = 0, \\ &\sum_{i=m}^{2m-1} c_i f'_{n_i}(b, \lambda) = 0, \\ &\quad\vdots \\ &\sum_{i=m}^{2m-1} c_i f_{n_i}^{(m-1)}(b, \lambda) = 0, \end{aligned} \tag{2.14}$$

for $c_m, c_{m+1}, \ldots, c_{2m-1}$. By Crammer's rule, we will get a nontrivial solution for the system (2.14) if

$$M_n(\lambda) = \begin{vmatrix} f_{n_m}(b, \lambda) & f_{n_{m+1}}(b, \lambda) & \ldots & f_{n_{2m-1}}(b, \lambda) \\ f'_{n_m}(b, \lambda) & f'_{n_{m+1}}(b, \lambda) & \ldots & f'_{n_{2m-1}}(b, \lambda) \\ \vdots & \vdots & \ldots & \vdots \\ f_{n_m}^{(m-1)}(b, \lambda) & f_{n_{m+1}}^{(m-1)}(b, \lambda) & \cdots & f_{n_{2m-1}}^{(m-1)}(b, \lambda) \end{vmatrix} = 0, \tag{2.15}$$

which is a polynomial in λ. Therefore the eigenvalues of the problem (1.1) are the roots of $M_n(\lambda)$. Section 2 may be summarized in the following algorithm.

Algorithm 2.1.

Step 1: Rewrite problem (1.1) in the format of Eq. (2.1).
Step 2: Use Eqs. (2.3) and (2.5), to define L and L^{-1}.
Step 3: Use Eq. (2.10) and initial conditions at $x = a$ to construct $y_0(x)$.
Step 4: Apply formula (2.11) to produce the sequence $\{y_n\}$ for some $K \in \mathbb{Z}^+$.
Step 5: Find roots of the polynomial (2.15), in which they are eigenvalues of problem (1.1).
Step 6: Find eigenfunctions $y_n(x)$ corresponding to eigenvalues λ_n for $n = 1, 2, ...$ by using (2.11).

3 Convergent Analysis

Convergence of the Adomian decomposition series solution was studied for different problems, (for example see [20], [21], [22], [23], [24], [25]). In present analysis we discuss the convergence properties of generalized MADM presented in Section 2 based on Banach fixed point theorem [26]. From (2.11), we obtain the successive approximation for the eigenfunctions of problem (1.1), where the exact solution can be derived from

$$y(x) = \lim_{n\to\infty} y_n(x). \tag{3.1}$$

Now, by using initial approximation y_0 (see (2.10)), the approximation solution can be considered by taking k-terms of the series (2.7), that is

$$y_k(x) = \sum_{i=0}^{k} y_i(x). \tag{3.2}$$

The modified Adomian decomposition method proposed in Section 2 makes a sequence $\{y_n\}$, here, we show that the sequence $\{y_n\}$ converges to the solution of problem (1.1). To do this, we state and prove the following theorems.

Theorem 3.1. *The series solution of problem (1.1) defined by (2.7) converges, if there exists $\alpha = CT$, $0 \le \alpha < 1$ such that $\|y_1\| < \infty$.*

Proof. Define the sequence $\{S_n\}_{n=0}^{\infty}$ as

$$\begin{aligned}
&S_0 = y_0,\\
&S_1 = y_0 + y_1,\\
&S_2 = y_0 + y_1 + y_2,\\
&\vdots\\
&S_n = y_0 + y_1 + \cdots + y_n,
\end{aligned} \tag{3.3}$$

and we show that $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space $H = L_w^2(a,b)$. We consider that

$$\|S_{n+1} - S_n\|_{L_w^2} = \|y_{n+1}\|_{L_w^2} \le \alpha \|y_n\|_{L_w^2} \le \cdots \le \alpha^{n+1} \|y_0\|_{L_w^2}. \tag{3.4}$$

Then for every $m \ge n$, we have

$$\begin{aligned}
\|s_m - s_n\|_{L_w^2} &\le \|s_{n+1} - s_n\|_{L_w^2} + \|s_{n+2} - s_{n+1}\|_{L_w^2} + \cdots + \|s_m - s_{m-1}\|_{L_w^2}\\
&\le \alpha^n[1 + \alpha + \cdots + \alpha^{m-n-1}]\|s_1 - s_0\|_{L_w^2}\\
&\le \tfrac{\alpha^n}{1-\alpha}\|y_1\|_{L_w^2}.
\end{aligned} \tag{3.5}$$

Since $\alpha \in (0,1)$, then $\|s_m - s_n\|_{L_w^2} \to 0$ as $m, n \to \infty$. Thus $\{s_n\}$ is a Cauchy sequence in the $L_w^2(a,b)$ space, therefore the series solution converges and the proof is complete. □

Theorem 3.2. *If the series solution (2.7) converges then it converges to the exact solution of the problem (1.1).*

Proof. For $y \in H = L_w^2(a,b)$, define an operator $\mathcal{L} : H \to H$ by

$$\mathcal{L}(y) = y_0(x) + L^{-1}F(y, y', \ldots, y^{(2m-2)}, \lambda) = y_0 + L^{-1}\sum_{n=0}^{\infty} A_n(x,\lambda). \tag{3.6}$$

Let $y, \hat{y} \in H = L^2_w(a,b)$, we have

$$\begin{aligned}
&\|\mathcal{L}(y)-\mathcal{L}(\hat{y})\|^2_{L^2_w} = \|L^{-1}R(y) - L^{-1}R(\hat{y})\|^2_{L^2_w} = \|L^{-1}R\,(y-\hat{y})\,\|^2_{L^2_w} \\
&= \int_a^b \Bigg| \underbrace{\int_0^x \int_0^{x_1} \cdots \int_0^{x_{m-1}}}_{m \text{ times}} \frac{1}{p_m(x_m)} \underbrace{\int_0^{x_m} \int_0^{x_{m+1}} \cdots \int_0^{x_{2m-1}}}_{m \text{ times}} R\,(y-\hat{y}) \cdot 1 dx_{2m} \ldots dx_1 \Bigg|^2 w(x)dx \\
&\le \int_a^b \Bigg(\underbrace{\int_0^x \int_0^{x_1} \cdots \int_0^{x_{m-1}}}_{m \text{ times}} \frac{1}{p_m(x_m)} \underbrace{\int_0^{x_m} \int_0^{x_{m+1}} \cdots \int_0^{x_{2m-1}}}_{m \text{ times}} \Big| R\,(y-\hat{y}) \Big|^2 dx_{2m} \ldots dx_1 \Bigg) \\
&\times \Bigg(\underbrace{\int_0^x \int_0^{x_1} \cdots \int_0^{x_{m-1}}}_{m \text{ times}} \frac{1}{p_m(x_m)} \underbrace{\int_0^{x_m} \int_0^{x_{m+1}} \cdots \int_0^{x_{2m-1}}}_{m \text{ times}} 1^2 dx_{2m} \ldots dx_1 \Bigg) w(x)dx \\
&\le K \int_a^b \Bigg(\underbrace{\int_0^x \int_0^{x_1} \cdots \int_0^{x_{m-1}}}_{m \text{ times}} \frac{1}{p_m(x_m)} \underbrace{\int_0^{x_m} \int_0^{x_{m+1}} \cdots \int_0^{x_{2m-1}}}_{m \text{ times}} \Big| R\,(y-\hat{y}) \Big|^2 dx_{2m} \ldots dx_1 \Bigg) w(x)dx \\
&\le K \underbrace{\int_0^x \int_0^{x_1} \cdots \int_0^{x_{m-1}}}_{m \text{ times}} \frac{1}{p_m(x_m)} \underbrace{\int_0^{x_m} \int_0^{x_{m+1}} \cdots \int_0^{x_{2m-1}}}_{m \text{ times}} \Big\| R\,(y-\hat{y}) \Big\|^2_{L^2_w} dx_{2m} \ldots dx_1 \\
&\le CT \|y-\hat{y}\|^2_{L^2_w} \le \alpha \|y-\hat{y}\|^2_{L^2_w},
\end{aligned}$$

where $\alpha = CT$ Therefore the mapping $\mathcal{L}$ is contraction and by the Banach fixed-point theorem for contraction [26], there is a unique solution of the problem (1.1). Now, we prove that the series solution (2.7) satisfies problem (1.1). It suffices to show that

$$L^{-1}R(y) = \lim_{n\to\infty} L^{-1}(N(S_n)) \tag{3.7}$$

Since Ny is Lipschitzian function, we have

$$\begin{aligned}
L^{-1}(R(y)) &= L^{-1}\left(R\left(\sum_{k=0}^{\infty} y_k\right)\right) \\
&= L^{-1}\left(R\left(\lim_{n\to\infty}\sum_{k=0}^{n} y_k\right)\right) \\
&= L^{-1}\left(R \lim_{n\to\infty} S_n\right) \\
&= \lim_{n\to\infty} L^{-1}(R(S_n)).
\end{aligned} \tag{3.8}$$

□

Theorem 3.3. *If the series solution (2.7) converges to the solution $y(x)$ and if the truncated series (3.2) is used as an approximation to the solution $y(x)$ for problem (1.1) then the error estimate is*

$$\left\| y(x) - \sum_{i=0}^{k} y_i \right\|_{L^2_w} \le \frac{\alpha^n}{1-\alpha} \|y_1\|_{L^2_w}. \tag{3.9}$$

Proof. From Theorem 3.1, we have

$$\|S_m - S_n\|_{L^2_w} \le \frac{\alpha^n}{1-\alpha} \|y_1\|_{L^2_w}, \quad m \ge n.$$

Now, when $m \to \infty$ then $S_m \to y(x)$. So

$$\|y(x) - S_n\|_{L^2_w} \le \frac{\alpha^n}{1-\alpha} \|y_1\|_{L^2_w}, \tag{3.10}$$

which implies that

$$\left\|y(x)-\sum_{i=0}^{k}y_i\right\|_{L_w^2}\leq\frac{\alpha^n}{1-\alpha}\|y_1\|_{L_w^2}. \tag{3.11}$$

This completes the proof. □

4 Numerical Results

In this section, we will apply the proposed algorithm to solve five high-order Sturm-Liouville problems. We are interested in approximating an eigenelement solution $(y(x),\lambda)$ to their corresponded eigenvalue problems.

Example 4.1. *Consider the following sixth-order Sturm-Liouville problem*

$$\begin{cases} -y^{(6)}(x)=\lambda y(x), \;\; x\in(0,\pi),\\ y(0)=y''(0)=y^{(4)}(0)=0,\\ y(\pi)=y''(\pi)=y^{(4)}(\pi)=0.\end{cases} \tag{4.1}$$

By using (2.10) and boundary conditions at $x=0$, we get

$$y_0(x)=c_1x+c_3\frac{x^3}{3!}+c_5\frac{x^5}{5!}$$

and using (2.11), we get

$$\begin{aligned}
y_1(x)&=\left(x-\lambda\frac{x^7}{7!}\right)c_1+\left(\frac{x^3}{3!}-\lambda\frac{x^9}{9!}\right)c_3+\left(\frac{x^5}{5!}-\lambda\frac{x^{11}}{11!}\right)c_5,\\
y_2(x)&=\left(x-\lambda\frac{x^7}{7!}+\lambda^2\frac{x^{13}}{13!}\right)c_1+\left(\frac{x^3}{3!}-\lambda\frac{x^9}{9!}+\lambda^2\frac{x^{15}}{15!}\right)c_3\\
&\quad+\left(\frac{x^5}{5!}-\lambda\frac{x^{11}}{11!}+\lambda^2\frac{x^{17}}{17!}\right)c_5,\\
y_3(x)&=\left(x-\lambda\frac{x^7}{7!}+\lambda^2\frac{x^{13}}{13!}-\lambda^3\frac{x^{19}}{19!}\right)c_1+\left(\frac{x^3}{3!}-\lambda\frac{x^9}{9!}+\lambda^2\frac{x^{15}}{15!}-\lambda^3\frac{x^{21}}{21!}\right)c_3\\
&\quad+\left(\frac{x^5}{5!}-\lambda\frac{x^{11}}{11!}+\lambda^2\frac{x^{17}}{17!}-\lambda^3\frac{x^{23}}{23!}\right)c_5,\\
&\vdots
\end{aligned} \tag{4.2}$$

More general, we see that

$$\begin{aligned}y_n(x,\lambda)&=\sum_{k=0}^{n}(-1)^k\lambda^k\frac{x^{6k+1}}{(6k+1)!}c_1+\sum_{k=0}^{n}(-1)^k\lambda^k\frac{x^{6k+3}}{(6k+3)!}c_3\\
&\quad+\sum_{k=0}^{n}(-1)^k\lambda^k\frac{x^{6k+5}}{(6k+5)!}c_5.\end{aligned} \tag{4.3}$$

Now, by applied Algorithm 1, the solution of (4.1) is

$$y(x,\lambda)=y_0(x,\lambda)+y_1(x,\lambda)+y_2(x,\lambda)+\cdots \tag{4.4}$$

Then by using n terms of (4.3) and boundary conditions at $x=\pi$, we get

$$\begin{vmatrix}
\sum_{i=0}^{n}(-\lambda)^i\frac{\pi^{(6i+1)}}{(6i+1)!} & \sum_{i=0}^{n}(-\lambda)^i\frac{\pi^{(6i+3)}}{(6i+3)!} & \sum_{i=0}^{n}(-\lambda)^i\frac{\pi^{(6i+5)}}{(6i+5)!}\\
\sum_{i=0}^{n}(-\lambda)^i\frac{\pi^{(6i-1)}}{(6i-1)!} & \sum_{i=0}^{n}(-\lambda)^i\frac{\pi^{(6i+1)}}{(6i+1)!} & \sum_{i=0}^{n}(-\lambda)^i\frac{\pi^{(6i+3)}}{(6i+3)!}\\
\sum_{i=0}^{n}(-\lambda)^i\frac{\pi^{(6i-3)}}{(6i-3)!} & \sum_{i=0}^{n}(-\lambda)^i\frac{\pi^{(6i-1)}}{(6i-1)!} & \sum_{i=0}^{n}(-\lambda)^i\frac{\pi^{(6i+1)}}{(6i+1)!}
\end{vmatrix}=0, \tag{4.5}$$

which is a polynomial in λ and roots of (4.5) are the eigenvalues of (4.1). The first sixth eigenvalues of problem (4.1) are given in Table 1. These results are convergence to exact solutions, for comparison results of present technique with other published papers in the literature (see for example [2], [6], [10]). Excellent agreements are observed between results of present technique and published papers. It is well known that the exact eigenvalues are given by $\lambda_k = k^6$ and the corresponding eigenfunction are $y_k = sin(kx)$.

Example 4.2. *Consider the following sixth-order Sturm-Liouville problem [10]*

$$\begin{cases} -y^{(6)}(x) + (3\alpha^2x^2y'')'' + ((8\alpha - 3\alpha^2x^4)y')' + (\alpha^3x^6 \\ \qquad\qquad -14\alpha^2x^2)y = \lambda y(x), \;\; x \in (0,\,5), \\ y(0) = y''(0) = y^{(4)}(0) = 0, \\ y(5) = y''(5) = y^{(4)}(5) = 0. \end{cases} \tag{4.6}$$

By using Algorithm 2.1, we get

$$\begin{aligned} y_0 &= c_1x + \frac{c_3}{6}x^3 + \frac{c_5}{120}x^5, \\ y_1 &= \left(x + \frac{1}{1235520}x^{13}\alpha^3 - \frac{13}{30240}x^9\alpha^2 - \frac{1}{5040}x^7\lambda\right)c_1 + \left(\frac{1}{6}x^3 + \frac{1}{21621600}\alpha^3x^{15} \right.\\ &\quad \left. - \frac{17}{498960}x^{11}\alpha^2 - \frac{1}{362880}x^9\lambda + \frac{13}{2520}x^7\alpha\right)c_3 + \left(\frac{1}{120}x^5 + \frac{1}{1069286400}\alpha^3x^{17}\right. \\ &\quad \left. - \frac{67}{74131200}x^{13}\alpha^2 - \frac{1}{39916800}x^{11}\lambda + \frac{17}{90720}x^9\alpha\right)c_5, \\ &\vdots \end{aligned} \tag{4.7}$$

The first three eigenvalues of problem (4.6) for $\alpha = 0.01$ are $\lambda_1 = 0.0997267782366864$, $\lambda_2 = 4.57232895602626$ and $\lambda_3 = 48.0416354201057$.

Example 4.3. *Consider the following eighth-order Sturm-Liouville problem*

$$\begin{cases} y^{(8)}(x) = \lambda y(x), \;\; x \in (0,\pi), \\ y(0) = y''(0) = y^{(4)}(0) = y^{(6)}(0) = 0, \\ y(\pi) = y''(\pi) = y^{(4)}(\pi) = y^{(6)}(\pi) = 0. \end{cases} \tag{4.8}$$

By using Algorithm 2.1, we get

$$\begin{aligned} y_0(x) &= c_1x + c_3\frac{x^3}{3!} + c_5\frac{x^5}{5!} + c_7\frac{x^7}{7!}, \\ y_1(x) &= \left(x + \lambda\frac{x^9}{9!}\right)c_1 + \left(\frac{x^3}{3!} + \lambda\frac{x^{11}}{11!}\right)c_3 + \left(\frac{x^5}{5!} + \lambda\frac{x^{13}}{13!}\right)c_5 + \left(\frac{x^7}{7!} + \lambda\frac{x^{15}}{15!}\right)c_7, \\ y_2(x) &= \left(x + \lambda\frac{x^9}{9!} + \lambda^2\frac{x^{17}}{17!}\right)c_1 + \left(\frac{x^3}{3!} + \lambda\frac{x^{11}}{11!} + \lambda^2\frac{x^{19}}{19!}\right)c_3 + \left(\frac{x^5}{5!} + \lambda\frac{x^{13}}{13!}\right. \\ &\quad \left. + \lambda^2\frac{x^{21}}{21!}\right)c_5 + \left(\frac{x^7}{7!} + \lambda\frac{x^{15}}{15!} + \lambda^2\frac{x^{23}}{23!}\right)c_7, \\ &\vdots \end{aligned} \tag{4.9}$$

In more general, we see that

$$y_n = \sum_{i=0}^{n}\lambda^i\frac{x^{(8i+1)}}{(8i+1)!}c_1 + \sum_{i=0}^{n}\lambda^i\frac{x^{(8i+3)}}{(8i+3)!}c_3 + \sum_{i=0}^{n}\lambda^i\frac{x^{(8i+5)}}{(8i+5)!}c_5 + \sum_{i=0}^{n}\lambda^i\frac{x^{(8i+7)}}{(8i+7)!}c_7. \tag{4.10}$$

By algorithm 2.1, the solution of problem (4.8) is

$$y(x,\lambda) = y_0(x,\lambda) + y_1(x,\lambda) + y_2(x,\lambda) + \cdots \tag{4.11}$$

and by using the boundary conditions at $x = \pi$ and n terms from (4.10), we will solve

$$\begin{vmatrix} \sum_{i=0}^{n} \lambda^i \frac{\pi^{(8i+1)}}{(8i+1)!} & \sum_{i=0}^{n} \lambda^i \frac{\pi^{(8i+3)}}{(8i+3)!} & \sum_{i=0}^{n} \lambda^i \frac{\pi^{(8i+5)}}{(8i+5)!} & \sum_{i=0}^{n} \lambda^i \frac{\pi^{(8i+7)}}{(8i+7)!} \\ \sum_{i=0}^{n} \lambda^i \frac{\pi^{(8i)}}{(8i)!} & \sum_{i=0}^{n} \lambda^i \frac{\pi^{(8i+1)}}{(8i+1)!} & \sum_{i=0}^{n} \lambda^i \frac{\pi^{(8i+3)}}{(8i+3)!} & \sum_{i=0}^{n} \lambda^i \frac{\pi^{(8i+3)}}{(8i+3)!} \\ \sum_{i=0}^{n} \lambda^i \frac{\pi^{(8i-3)}}{(8i-3)!} & \sum_{i=0}^{n} \lambda^i \frac{\pi^{(8i-1)}}{(8i-1)!} & \sum_{i=0}^{n} \lambda^i \frac{\pi^{(8i+1)}}{(8i+1)!} & \sum_{i=0}^{n} \lambda^i \frac{\pi^{(8i+3)}}{(8i+3)!} \\ \sum_{i=0}^{n} \lambda^i \frac{\pi^{(8i-5)}}{(8i-5)!} & \sum_{i=0}^{n} \lambda^i \frac{\pi^{(8i-3)}}{(8i-3)!} & \sum_{i=0}^{n} \lambda^i \frac{\pi^{(8i-1)}}{(8i-1)!} & \sum_{i=0}^{n} \lambda^i \frac{\pi^{(8i+1)}}{(8i+1)!} \end{vmatrix} = 0, \tag{4.12}$$

which is a polynomial in λ. By computing roots of (4.12), we can obtain the eigenvalues of problem (4.8). The first six eigenvalues are listed in Table 1.

Example 4.4. *Consider the following tenth-order Sturm-Liouville problem*

$$\begin{cases} -y^{(10)}(x) = \lambda y(x), \ \ x \in (0, \pi), \\ y(0) = y''(0) = y^{(4)}(0) = y^{(6)}(0) = y^{(8)}(0) = 0, \\ y(\pi) = y''(\pi) = y^{(4)}(\pi) = y^{(6)}(\pi) = y^{(8)}(\pi) = 0. \end{cases} \tag{4.13}$$

Now, by applied Algorithm 2.1, we have

$$\begin{aligned} y_0(x) &= c_1 x + c_3 \frac{x^3}{3!} + c_5 \frac{x^5}{5!} + c_7 \frac{x^7}{7!} + c_9 \frac{x^9}{9!}, \\ y_1(x) &= \left(x - \lambda \frac{x^{11}}{11!}\right) c_1 + \left(\frac{x^3}{3!} - \lambda \frac{x^{13}}{13!}\right) c_3 + \left(\frac{x^5}{5!} - \lambda \frac{x^{15}}{15!}\right) c_5 + \left(\frac{x^7}{7!} - \lambda \frac{x^{17}}{17!}\right) c_7 \\ &\quad + \left(\frac{x^9}{9!} - \lambda \frac{x^{19}}{19!}\right) c_9, \\ y_2(x) &= \left(x - \lambda \frac{x^{11}}{11!} + \lambda^2 \frac{x^{21}}{21!}\right) c_1 + \left(\frac{x^3}{3!} - \lambda \frac{x^{13}}{13!} + \lambda^2 \frac{x^{23}}{23!}\right) c_3 + \left(\frac{x^5}{5!} - \lambda \frac{x^{15}}{15!} \right. \\ &\quad \left. + \lambda^2 \frac{x^{25}}{25!}\right) c_5 + \left(\frac{x^7}{7!} - \lambda \frac{x^{17}}{17!} + \lambda^2 \frac{x^{27}}{27!}\right) c_7 + \left(\frac{x^9}{9!} - \lambda \frac{x^{19}}{19!} + \lambda^2 \frac{x^{29}}{29!}\right) c_9, \\ &\vdots \end{aligned} \tag{4.14}$$

We see that

$$\begin{aligned} y_n(x, \lambda) &= \sum_{i=0}^{n} (-\lambda)^i \frac{x^{(10i+1)}}{(10i+1)!} c_1 + \sum_{i=0}^{n} (-\lambda)^i \frac{x^{(10i+3)}}{(10i+3)!} c_3 \\ &\quad + \sum_{i=0}^{n} (-\lambda)^i \frac{x^{(10i+5)}}{(10i+5)!} c_5 + \sum_{i=0}^{n} (-\lambda)^i \frac{x^{(10i+7)}}{(10i+7)!} c_7 \\ &\quad + \sum_{i=0}^{n} (-\lambda)^i \frac{x^{(10i+9)}}{(10i+9)!} c_9. \end{aligned} \tag{4.15}$$

Now by using the boundary conditions at $x = \pi$ and n terms from (4.15), we will solve

$$\begin{vmatrix} \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i+1)}}{(10i+1)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i+3)}}{(10i+3)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i+5)}}{(10i+5)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i+7)}}{(10i+7)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i+9)}}{(10i+9)!} \\ \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i-1)}}{(10i-1)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i+1)}}{(10i+1)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i+3)}}{(10i+3)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i+5)}}{(10i+5)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i+7)}}{(10i+7)!} \\ \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i-3)}}{(10i-3)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i-1)}}{(10i-1)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i+1)}}{(10i+1)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i+3)}}{(10i+3)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i+5)}}{(10i+5)!} \\ \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i-5)}}{(10i-5)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i-3)}}{(10i-3)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i-1)}}{(10i-1)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i+1)}}{(10i+1)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i+3)}}{(10i+3)!} \\ \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i-7)}}{(10i-7)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i-5)}}{(10i-5)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i-3)}}{(10i-3)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i-1)}}{(10i-1)!} & \sum_{i=0}^{n}\frac{(-\lambda)^i\pi^{(10i+1)}}{(10i+1)!} \end{vmatrix} = 0, \tag{4.16}$$

which is a polynomial in λ. The roots of (4.16) are eigenvalues of problem (4.13). The first sixth eigenvalues are computed and listed in Table 1.

Example 4.5. *Consider the following fourth-order Sturm-Liouville problem related to mechanicals non-linear systems identification [7], [10], [14]*

$$\begin{cases} y^{(4)}(x) - 2\alpha x^2 y'' - 4\alpha x y' + (\alpha^2 x^4 - 2\alpha)y = \lambda y(x), \quad x \in (0,5), \\ y(0) = y''(0) = 0, \\ y(5) = y''(5) = 0. \end{cases} \tag{4.17}$$

By using Algorithm 2.1, we have

$$\begin{aligned} y_0(x) &= c_1 x + c_3\frac{x^3}{3!} \\ y_1(x) &= \left(x - \frac{1}{3024}\alpha^2 x^9 + \frac{1}{20}x^5\alpha + \frac{1}{120}x^5\lambda\right)c_1 + \left(\frac{1}{6}x^3 - \frac{1}{47520}\alpha^2 x^{11}\right. \\ &\quad \left. + \tfrac{13}{2520}x^7\alpha + \tfrac{1}{5040}x^7\lambda\right)c_3 \\ y_2(x) &= \left(x + \frac{1}{172730880}\alpha^4 x^{17} - \frac{131}{259459200}x^{13}\lambda\alpha^2 - \frac{119}{18532800}x^{13}\alpha^3\right. \\ &\quad \left. + \frac{1}{1440}\alpha^2 x^9 + \frac{17}{90720}x^9\alpha\lambda + \frac{1}{362880}x^9\lambda^2 + \frac{1}{120}x^5\alpha + \frac{1}{120}x^5\lambda\right)c_1 \\ &\quad + \left(\frac{1}{6}x^3 - \frac{1}{4420500480}\alpha^4 x^{19} + \frac{73}{5448643200}x^{15}\alpha^3 - \frac{59}{10897286400}x^{15}\lambda\alpha^2\right. \\ &\quad + \frac{59}{1108800}\alpha^2 x^{11} + \frac{1}{285120}x^{11}\alpha\lambda + \frac{1}{39916800}x^{11}\lambda^2 + \frac{13}{2520}x^7\alpha \\ &\quad \left. + \tfrac{1}{5040}x^7\lambda\right)c_3 \\ &\ \vdots \end{aligned}$$

The first three eigenvalues of problem (4.17), for $\alpha = 0.01$ are: $\lambda_1 = 0.21505086447024$, $\lambda_2 = 2.75480992983924$ and $\lambda_3 = 13.21535155405568$.

5 Conclusion

Present paper exhibits the applicability of the modified Adomian decomposition method to solve high-order Sturm-Liouville eigenvalue problems. In this work we prove that proposed method is convergent and is well suited to solve high-order Sturm-Liouville problems. Numerical results obtained by using the modified Adomian decomposition method described in Section 2 show excellent agreement with the exact solution when one uses only a few terms.

Competing Interests

The authors declare that no competing interests exist.

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Table 1: The first six eigenvalues for Examples 1, 3, 4.

Ex.	k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
	2	1.000012982591777	56.278890029834861				
	3	0.999999999669759	64.019803409965889	532.149383877359631			
	4	1.000000000000002	63.999993955805054	731.361440711892019	2356.075545405737775		
	5	1.000000000000000	64.000000000480616	728.997819007874182	4228.049464086881536		
1	6	1.000000000000000	63.999999999999987	729.000000676691570	4095.803602772923499		
	7	1.000000000000000	64.000000000000000	728.999999999917061	4096.000137789611629		
	8	1.000000000000000	64.000000000000000	729.000000000000005	4095.999999956884880	15625.010847562816665	46394.783558877924645
	9	1.000000000000000	64.000000000000000	729.000000000000000	4096.000000000006639	15624.999993556946824	46656.493105853629153
	10	1.000000000000000	64.000000000000000	729.000000000000000	4095.999999999999999	15625.000000002026883	46655.999532344118299
	11	1.000000000000000	64.000000000000000	729.000000000000000	4096.000000000000000	15625.000000000000000	46655.999999999921245
	12	1.000000000000000	64.000000000000000	729.000000000000000	4096.000000000000000	15625.000000000000000	46656.000000000000015
	13	1.000000000000000	64.000000000000000	729.000000000000000	4096.000000000000000	15625.000000000000000	46656.000000000000000
	2	1.0000000139424631	255.1912484516522934				
	3	1.0000000000000015	255.9999758252043787	6582.1556126715761096			
	4	1.0000000000000000	255.9999999999357797	6561.0014872300863400	65301.5245144208390651		
	5	1.0000000000000000	256.0000000000000000	6561.0000000165164423	65535.9727164431913067	392431.0850276472580053	
	6	1.0000000000000000	256.0000000000000000	6561.0000000000000410	65535.9999993102692247	390625.2710159350685007	1670123.8569155902998318
3	7	1.0000000000000000	256.0000000000000000	6561.0000000000000000	65535.9999999999950702	390625.0000117142127737	1679614.1356936972701543
	8	1.0000000000000000	256.0000000000000000	6561.0000000000000000	65536.0000000000000000	390625.0000000001698410	1679615.9998824062972465
	9	1.0000000000000000	256.0000000000000000	6561.0000000000000000	65536.0000000000000000	390625.0000000000000009	1679615.9999999971653998
	10	1.0000000000000000	256.0000000000000000	6561.0000000000000000	65536.0000000000000000	390625.0000000000000000	1679615.9999999999999710
	11	1.0000000000000000	256.0000000000000000	6561.0000000000000000	65536.0000000000000000	390625.0000000000000000	1679616.0000000000000000
	12	1.0000000000000000	256.0000000000000000	6561.0000000000000000	65536.0000000000000000	390625.0000000000000000	1679616.0000000000000000
	1	0.9997297302962838					
	2	1.0000000000064805	1023.9747265452897512				
	3	1.0000000000000000	1024.0000000076896544	59048.8233931585398257			
	4	1.0000000000000000	1023.9999999999999999	59049.0000001486483621	1048575.5634512260551001		
	5	1.0000000000000000	1024.0000000000000000	59048.9999999999999880	1048576.0000006373640029	9765624.3859898835405941	
4	6	1.0000000000000000	1024.0000000000000000	59049.0000000000000000	1048575.9999999999998610	9765625.0000012665779169	60466175.3942395260829979
	7	1.0000000000000000	1024.0000000000000000	59049.0000000000000000	1048576.0000000000000000	9765624.9999999999994714	60466176.0000015840152622
	8	1.0000000000000000	1024.0000000000000000	59049.0000000000000000	1048576.0000000000000000	9765625.0000000000000000	60466175.9999999999989557
	9	1.0000000000000000	1024.0000000000000000	59049.0000000000000000	1048576.0000000000000000	9765625.0000000000000000	60466176.0000000000000000
	10	1.0000000000000000	1024.0000000000000000	59049.0000000000000000	1048576.0000000000000000	9765625.0000000000000000	60466176.0000000000000000