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Closed-Form Equation for the Duration of Daily Insolation on Uniformly Sloping Terrain

Hugo A. Loáiciga[1*]

[1]*Department of Geography University of California Santa Barbara California 93106, USA.*

Author's contribution

The only author performed the whole research work. Author HAL wrote the first draft of the paper. Author HAL read and approved the final manuscript.

Research Article

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ABSTRACT

Aims: Derivation of a closed-form expression for the duration of the daily insolation on surfaces of arbitrary uniform slope and aspect located anywhere on Earth, anytime of the year.
Study Method: It is shown that the sunrise- and sunset-hour angles and the duration of daily insolation depend on the roots of the equation $A \cos\phi + B \sin\phi + C = 0$, in which ϕ is the hour angle and A, B, and C are coefficients that involve the slope of the surface, the aspect of the sloping surface, the solar declination, and the latitude of a point of interest on the sloping surface.
Results: The method to calculate the duration of daily insolation developed in this article is applicable to any sequence or combinations of days to obtain the total number of daylight hours over arbitrary periods. Solutions for double sunrise and double sunset situations are also derived in this paper.
Conclusion: The closed-form equations developed in this paper can be used in conjunction with measurements of atmospheric transmissivity to calculate the direct solar insolation on a surface of arbitrary slope and aspect, yielding a powerful tool for agricultural, meteorologic, hydrologic, ecologic, and climatic studies and modeling.

Keywords: Solar radiation; slope; aspect; energy balance; evapotranspiration.

**Corresponding author: Email: hugo@geog.ucsb.edu;*

1. INTRODUCTION

The duration of daily insolation is a prime determining factor of the energy input to the Earth's surface. The duration and intensity of solar-energy input to the earth's surface drive evaporation [1], evapotranspiration [2,3,4], photosynthesis [5], snowmelt [6], soil and surface-air heating [7,8], they influence microclimates [9], the crop response to solar insolation [10], biophysical ecology [11], and play a central role on the overall earth-atmosphere radiation balance [12]. The Blaney-Criddle method [2], used to calculate the evapotranspiration by crops, and which has been widely applied, has as one of its input variables the duration of clear-sky insolation during the calculation period [13]. The duration and intensity of solar insolation are variables with primordial roles in agricultural meteorology.

This paper focuses on the determination of the duration of daily insolation on uniformly sloping terrain. The main objective of this work is to find a closed-form equation to calculate the times of sunrise and sunset in terrain of arbitrary (uniform) slope and aspect at any latitude and on any day of the year. The closed-form equation is solved numerically and avoids the more involved use of the equivalent slope method [14]. These two features – expediency and improved accuracy in the calculation of daily insolation- are novelties of the method presented in this paper relative to previous related work. The usefulness of a closed-form equation for daily insolation is evident from its centrality in determining the solar energy input to sloping terrain, which, in turn, is a key factor controlling biophysical terrestrial processes [15]. The calculation of the daily solar radiation to sloping terrain is possible once the sunrise and sunset angles are determined from the results of this paper, and in combination with atmospheric properties as explained in the subsection The daily solar radiation input to sloping terrain. This paper provides a cost-effective computational alternative to proprietary software with solar-radiation calculation capabilities. The paper's method can be merged with non-proprietary, open-access, software for calculation of solar radiation input to sloping terrain (for example, SolarRad). In addition, it extends the work of previous authors concerning the calculation of solar radiation input to terrain of arbitrary slope, aspect, for any latitude and solar declination [8,15,16,17,18].

The method presented in this work can be applied to any sequence or combination of days to yield the total daylight hours during an arbitrary period. Calculating the duration of insolation on sloping terrain requires the exact determination of the geometry of (direct) solar radiation reaching a sloping surface. That determination is facilitated by first examining the geometry of insolation on a horizontal surface, which is undertaken next.

2. METHODOLOGIES

2.1 Insolation on a Horizontal Surface

2.1.1 Geometric fundamentals

Fig. 1 depicts the beam of direct solar radiation (**b**) reaching a point P on horizontal terrain. The point P is located uniquely by the latitude θ and hour angle ϕ. A positive (or negative) hour angle is measured counterclockwise (or clockwise) from the solar-noon meridian to the meridian containing the point P. The hour angle is depicted by the circular sector a-P'-P on a plane parallel to the equatorial plane in Fig. 1. The temporal rate of change of hour angle ϕ

is $\dot{\phi}$ = dϕ/dt = d(ωt)/dt = ω, in which ω is the Earth's rotational angular velocity (approximately 2π radians / 24 hr, and counterclockwise when seeing the Earth over the north pole, where 1 radian = 180/π angular degrees). Time t = 0 is chosen to correspond to solar noon, and the hour angle equals zero along the solar-non meridian. In other words, the hour angle evaluated at time zero is $\phi = \omega t = \omega\ 0 = 0$. The beam of direct solar radiation strikes perpendicular to a horizontal surface at point Z' on the solar-noon meridian. The latitude of point Z', that is, the latitude at which the sun is directly overhead at solar noon, is called the solar declination (δ). Its range is approximately $-23.45° \leq \delta \leq 23.45°$ [19], being positive (or negative) when it is a northern (or southern) latitude. The solar declination is about $+23.45°$ ($-23.45°$) on summer (or winter) solstice in the northern hemisphere, and equals zero on the autumnal and vernal equinoxes. The solar declination (δ) is approximated by the following equation ([13]:

$$\delta = 23.45 \sin\left[360\left(284 + \mathrm{J}\right)/365\right] \qquad (1)$$

in which δ and the argument of the sine function are in degrees, and J is the day of the year (J = 1 on January 1st at midnight , J = 365 on December 31st at midnight, etc.). Fig. 1 also shows the approximate $23.45°$ tilt of the Earth's rotation axis (N-S) with respect to the line-segment O-O' perpendicular to the plane of the ecliptic. The latter plane contains the orbit followed by the Earth as it revolves about the sun.

2.1.2 The daily solar radiation input to horizontal terrain

The relation between the duration of daily insolation and the daily energy input by direct solar radiation on a horizontal surface ($\mathrm{I_H}$, J m^{-2}) is embodied by the following equation:

$$I_H = \int_{\phi_{sr0}}^{\phi_{ss0}} \tau(\phi)\ \varepsilon I_0 \cos\zeta_0\ d\phi \qquad (2)$$

in which: I_0 is the solar radiation flux at the top of the atmosphere at the mean Earth-sun distance (I_0, the solar constant, is about 1367 W m^{-2} = 1.181 x 10^{8} J m^{-2} d^{-1}); ε, the eccentricity ratio, equals $(r_0 / r)^2$, where r_0 is the Earth-sun mean distance and *r* is the Earth-sun distance on any particular day of the year; ζ_0 is the angle comprised between the beam vector (**b**) of direct solar radiation impinging at a specified point P (with latitude θ and hour angle ϕ) on the horizontal surface and a line perpendicular to the horizontal surface at P ($\zeta_0 = \cos^{-1}(-\cos\delta\ \cos\theta\ \cos\phi - \sin\delta \sin\theta)$, in radians); ϕ_{sr0} and ϕ_{ss0} are the sunrise-hour and sunset-hour angles (expressed in radians in the limits of integration of equation (2)), respectively, which depend on the latitude θ and solar declination δ; τ is the (total) atmospheric transmissivity ($0 < \tau < 1$, [20,21,22]). The transmissivity depends on the hour angle in equation (2) in a manner that requires numerical integration of its right-hand side (see, for example, [22]). The radiative flux |**b**| = $\tau\,\varepsilon\, I_0$ equals the fraction of the solar constant that is transmitted to the Earth's surface by the atmosphere. The integrand on the right-hand side of equation (2) is a special case of Lambert's cosine law, which states that the radiative flux received by a surface equals the component of the beam of solar radiation perpendicular to the surface. This component is precisely $\tau\, \varepsilon\, I_0 \cos\zeta_0$. The integral in equation (2) is over the range of the hour angle during which there is insolation on the horizontal surface at point P. This description of the key geometric elements governing the insolation of a horizontal surface is generalized in the next section to the case of sloping terrain.

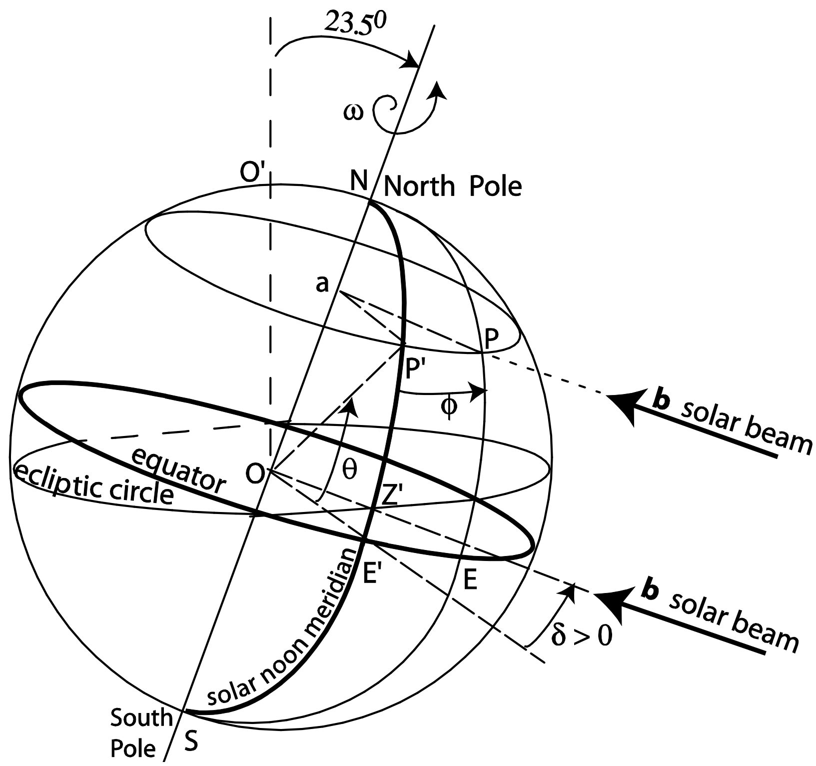

Fig. 1. Geometry and variables involved in the determination of the duration of daily insolation and energy input by solar radiation. $\delta, \theta, \phi, \omega$ denote the solar declination, latitude, hour angle, and the Earth's angular rotational velocity, respectively

3. THE GEOMETRY OF INSOLATION ON A SLOPING SURFACE

Sunrise (or sunset) occurs when solar radiation shines for the first (or last) time upon a surface assuming clear-sky conditions in any given day. The duration of daily insolation equals the time of sunset minus the time of sunrise. There may be double times of sunrise and sunset for certain combinations of slope, aspect, latitude, and solar declination, in which case the determination of the duration of daily insolation becomes more involved. In some instances daily insolation on a sloping surface (or on level terrain) may last 24 hours. The following sections derive the times of sunrise and sunset in terrain of arbitrary slope and aspect at any latitude and on any day of the year. It is assumed in this work that insolated areas are not shaded by topographic promontories or other obstacles and that the slope of insolated terrain is uniform. Terrain of variable slope can be approached by applying the method of this paper to a series of adjacent, short, portions of the terrain with unequal slopes that approximate the actual shape of the land.

3.1 Spherical Coordinate System

The determination of the duration of daily insolation on sloping terrain is helped by introducing a spherical coordinate system that is used to capture the passage from a horizontal to a sloping surface. The spherical coordinate system features three mutually orthogonal unit direction vectors (e_r, e_θ, and e_ϕ), shown in Fig. 2, which depicts an oblique view of the solar-noon meridian and of the meridian containing the point P of latitude θ and hour angle ϕ.

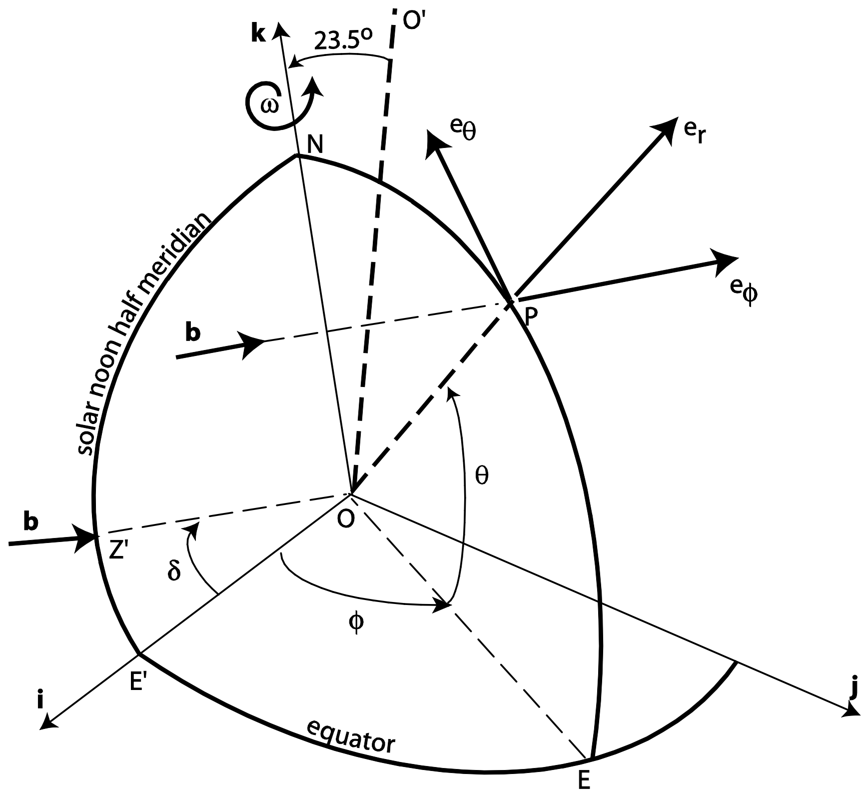

Fig. 2. Oblique view of coordinate systems. O is the Earth's center, N the North Pole, b the vector of direct solar radiation reaching the Earth's surface, ϕ and θ are the hour angle and latitude of the meridian containing point P, respectively, ω the Earth's angular rotational velocity; e_r, e_θ, and e_ϕ are the unit direction vectors associated with the spherical coordinate system; i, j, and k are the unit vectors of the Cartesian coordinate system

The point P lies on the sloping surface under study. Notice that the latitude is positive (or negative) north (or south) of the equator. The unit vector e_r is directed radially outward at point P and is perpendicular to the horizontal plane tangential at point P. e_θ points in the direction of increasing absolute value of the latitude and is tangential to the meridian containing point P at point P. e_ϕ points in the direction of increasing hour angle and is perpendicular to e_θ and e_r. The unit vectors so defined can be related to a Cartesian system of coordinates defined in terms of (mutually orthogonal) unit direction vectors **i**, **j**, and **k** with origin at the Earth's center (point O in Fig. 2).

The **k** axis in Fig. 2 coincides with the direction of the line segment O-N, which is part of the Earth's rotation axis. The unit vectors **i** and **j** lie on the equatorial plane, with the vector **i** corresponding to an hour angle $\phi = 0$. The e_r, e_θ, and e_ϕ unit vectors can be expressed as follows as a function of the latitude, hour angle, and the Cartesian unit vectors **i**, **j**, and **k**:

$$e_r = \cos\theta \cos\phi \;\, \boldsymbol{i} + \cos\theta \sin\phi \;\, \boldsymbol{j} + \sin\theta \;\, \boldsymbol{k} \tag{3}$$

$$e_\theta = -\sin\theta \cos\phi \;\, \boldsymbol{i} - \sin\phi \sin\theta \;\, \boldsymbol{j} + \cos\theta \;\, \boldsymbol{k} \tag{4}$$

$$e_\phi = -\sin\phi \;\, \boldsymbol{i} + \cos\phi \;\, \boldsymbol{j} \tag{5}$$

The unit vectors e_r, e_θ, and e_ϕ in equations (3)-(5) are consistent with a horizontal slope at point P. The next subsection shows how to convert them into a coordinate system consistent with a sloping surface.

3.1.1 Coordinate system for a sloping surface

Fig. 3 shows how a plane can be rotated Ω degrees west (or east) of north to achieve aspects ranging from 0° degrees west (or east) of north to 180° west (or east) of north, where Ω is the angle of orientation of a slope, or aspect. The convention of this paper is to make Ω positive if the rotation is west of north, or negative if the rotation is east of north. The rotation shown in Fig. 3 is with respect to the radial unit vector e_r (Fig. 2), to obtain rotated unit vectors $e'_r = e_r$, $e'_\theta = \cos\Omega \;\, e_\theta - \sin\Omega \;\, e_\phi$, and $e'_\phi = \sin\Omega \;\, e_\theta c + \cos\Omega \;\, e_\phi$.

Fig. 4 depicts a view perpendicular to the great circle containing the solar-noon meridian. This simplified, 2-dimensional, view shows several of the geometric factors governing the insolation of a sloping surface. For simplicity, the aspect Ω = 0 in Fig. 4. Point P' is at the base of a slope. A slope can be downward or upward from the horizontal plane tangential at P', where the latter plane contains the unit vector e'_θ (and e'_ϕ, as well, which is hidden by the 2-dimensional perspective used). This paper's convention is to assign a positive sign to a downward slope, or a negative sign to an upward slope.

The downward slope P'-1 (or upward P'-2) shown in Fig. 4 is obtained by rotating the unit vector e_θ counterclockwise α (or clockwise α_1) degrees, as seen in Fig. 4. The axis of rotation in this case is the (hidden) unit vector e'_ϕ which is aimed onto and perpendicular to the plane of Fig. 4. The downward slope in Fig. 4 was rotated α degrees, which in this case is the critical angle that, if exceed, would result in a shaded slope during solar noon.

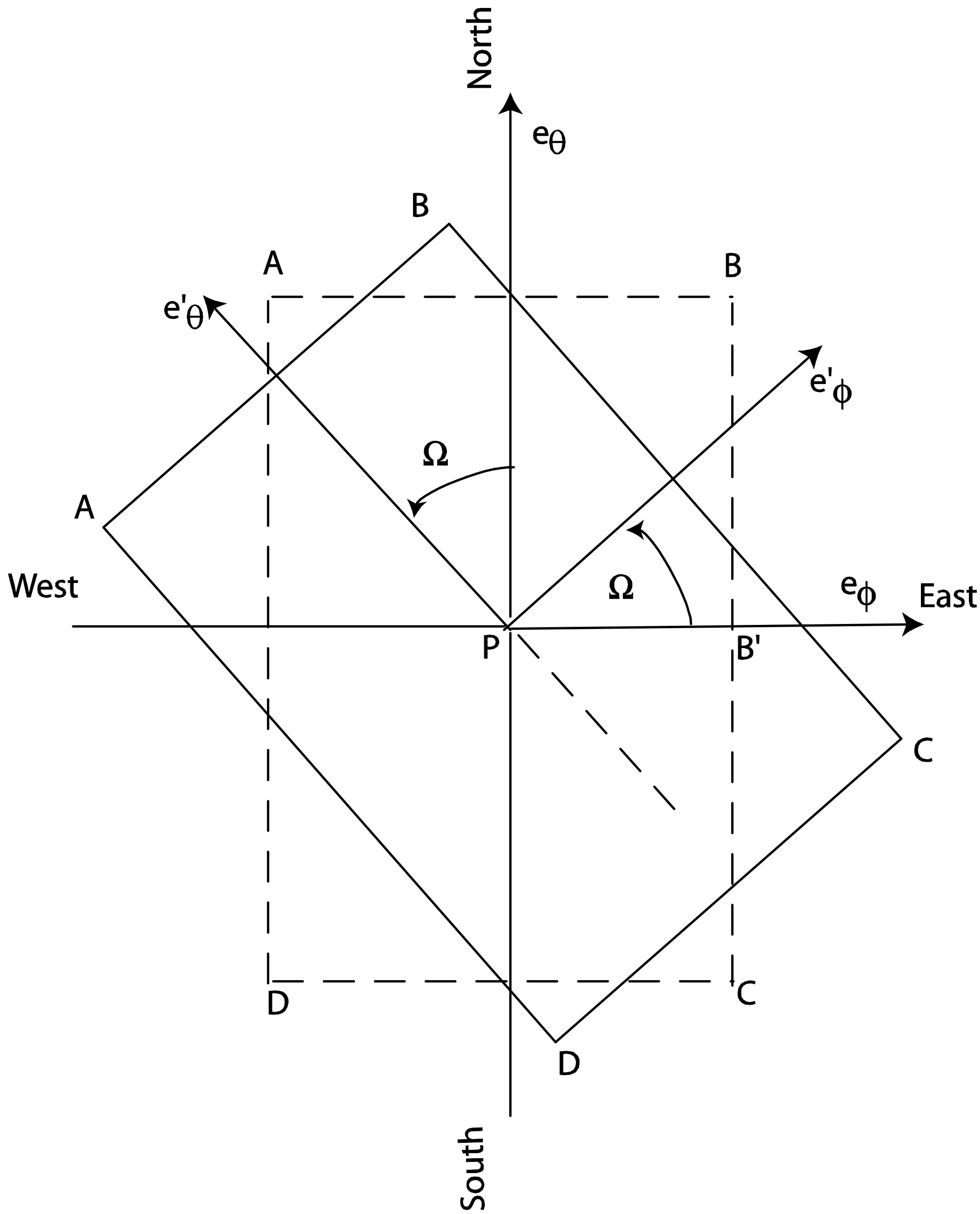

Fig. 3. Rotation of the spherical coordinate system to achieve a desired aspect Ω. The axis of rotation coincides with the unit vector e_r

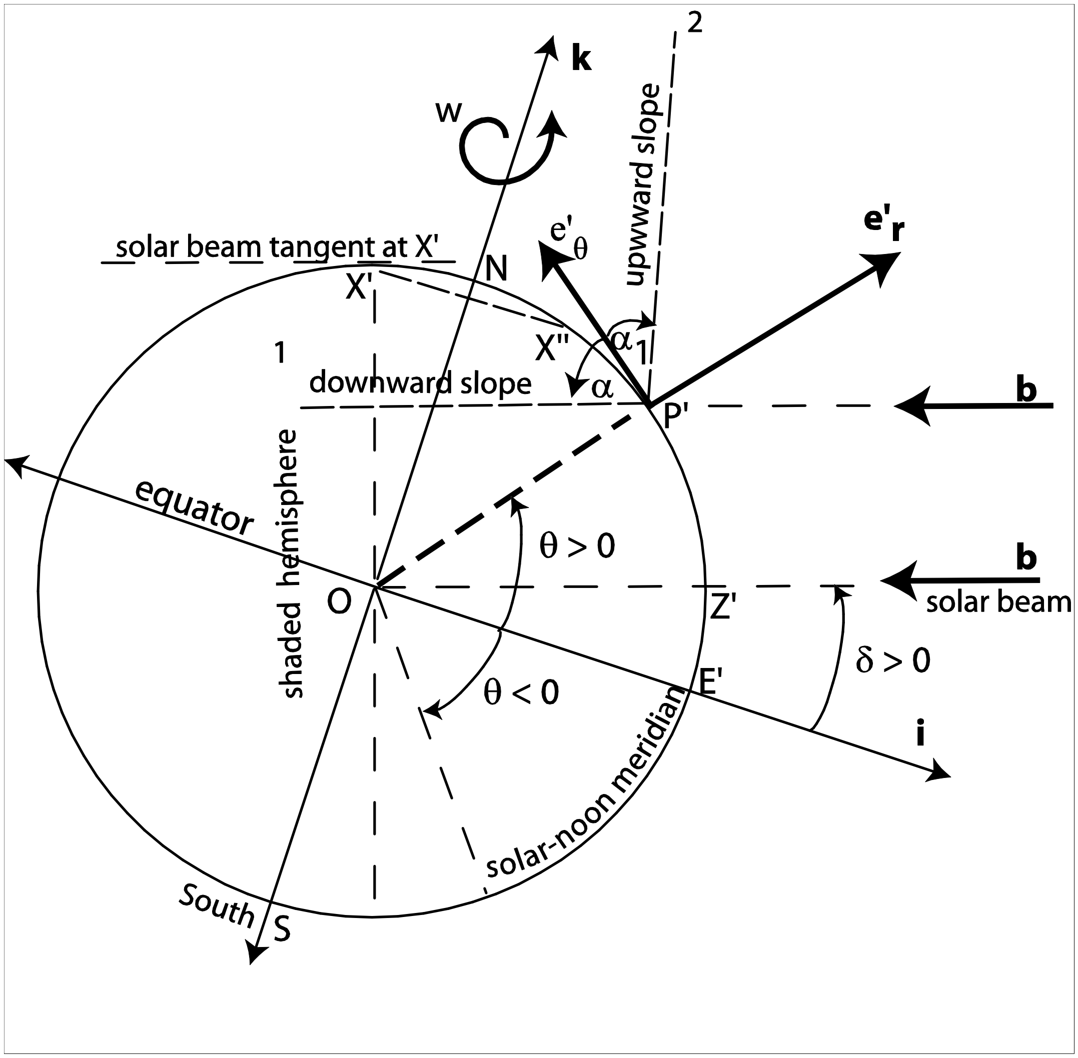

Fig. 4. View perpendicular to the plane of great circle containing the solar-noon meridian illustrating the relation of the solar beam to key geometric variables. An aspect $\Omega = 0$ is assumed in the illustration. α_1 and α are upward and downward slopes, respectively; θ, δ, and ω denote latitude, solar declination, and the Earth's angular rotational velocity, respectively; O and N are the Earth's center and North Pole, respectively. X'' receives 24-hr daily insolation. The shaded hemisphere is leftward of the axis containing the points O and X'

It is pertinent in this analysis to highlight that locations of high latitude, such as X'' on Fig. 4, receive 24-hr daily insolation for the shown solar declination. In fact, as shown on Fig. 4, any location with latitude higher than that of X'' is lighted 24 hours daily. For specifics, if the view in Fig. 4 represented the summer solstice in the northern hemisphere then, the solar declination would be $\delta = 23.45°$, and day-long insolation would be experienced at latitudes $66.55° \leq \theta \leq 90°$.

Fig. 5 is a graphical summary of a rotation of the coordinate vectors e'_r, e'_θ, and e'_ϕ exerted to achieve a downward (that is, positive) slope. The doubly-rotated coordinates vectors e''_r and e''_θ, are shown in Fig. 5. The third unit vector $e''_\phi = e'_\phi$ is perpendicular onto the plane of Fig. 5 and serves as the axis of rotation in this instance.

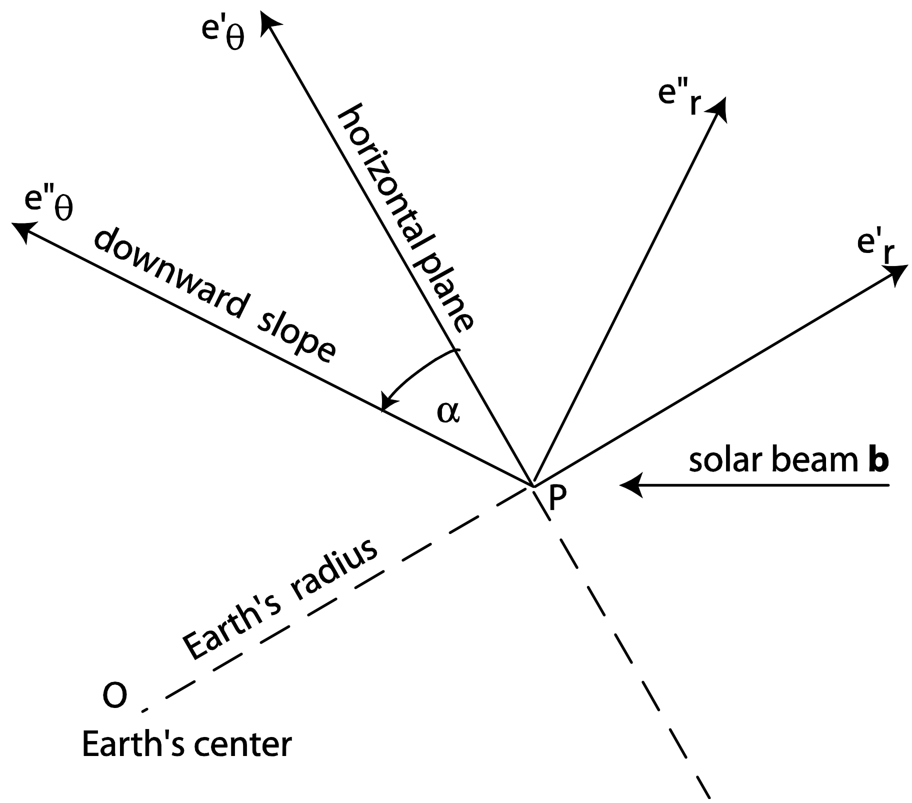

Fig. 5. Rotation of the spherical coordinate system to achieve a desired slope α. The axis of rotation coincides with the unit vector e'_ϕ

An upward (that is, negative) slope would be achieved by making the rotation shown in Fig. 5 in a clockwise direction. The doubly-rotated unit vectors e''_r, e''_θ, and e''_ϕ provide the coordinate system with which to describe the geometry of a sloping surface in full generality, and are given by the following equations:

$$\begin{aligned} e''_r = &\left[\cos\alpha\cos\theta\cos\phi - \sin\alpha\cos\Omega\sin\theta\cos\phi + \sin\alpha\sin\Omega\sin\phi\right]\boldsymbol{i} + \\ &\left[\cos\alpha\cos\theta\sin\phi - \sin\alpha\cos\Omega\sin\theta\sin\phi - \sin\alpha\sin\Omega\cos\phi\right]\boldsymbol{j} + \\ &\left[\cos\alpha\sin\theta + \sin\alpha\cos\Omega\cos\theta\right]\boldsymbol{k} \end{aligned} \tag{6}$$

$$e''_{\theta} = \left[-\sin\alpha\cos\theta\cos\phi - \cos\alpha\cos\Omega\sin\theta\cos\phi + \cos\alpha\sin\Omega\sin\phi\right]\boldsymbol{i} + \\ -\left[\sin\alpha\cos\theta\sin\phi + \cos\alpha\cos\Omega\sin\theta\sin\phi + \cos\alpha\sin\Omega\cos\phi\right]\boldsymbol{j} + \\ \left[-\sin\alpha\sin\theta + \cos\alpha\cos\Omega\cos\theta\right]\boldsymbol{k} \tag{7}$$

$$e''_{\phi} = -\left[\cos\Omega\ \sin\phi + \sin\Omega\sin\theta\cos\phi\right]\boldsymbol{i} + \\ \left[\cos\Omega\cos\phi - \sin\Omega\sin\theta\sin\phi\right]\boldsymbol{j} + \\ \left[\sin\Omega\cos\theta\right]\boldsymbol{k} \tag{8}$$

The unit vectors in equations (6), (7), (8) revert to those in equations (3), (4), (5), respectively, when the slope angle α and the aspect Ω equal zero.

3.1.2 The daily solar radiation input to sloping terrain

By analogy with equation (2) and considering the geometry presented in Fig. 1, the daily energy input by solar radiation to a sloping surface (I_S, in J m^{-2}) is given by the following expression:

$$I_S = \int_{\phi_{sr}}^{\phi_{ss}} \tau(\phi)\,\varepsilon\, I_0\,\cos\zeta\; d\phi \tag{9}$$

in which I_0, ε, ϕ, and τ were defined after equation (2); ϕ_{sr} and ϕ_{ss} are the sunrise- and sunset-hour angles, respectively, at point P, where the sloping surface may exhibit downward (or upward) descent (or ascent); ζ is the angle comprised between the beam of direct solar radiation impinging upon point P on the sloping surface and a line normal to the horizontal surface at P, which can be shown to be given (in radians) by:

$$\zeta = \cos^{-1}\left(A\cos\phi + B\sin\phi + C\right) \tag{10}$$

where:

$$A = -\cos\delta\cos\alpha\cos\theta + \cos\delta\sin\alpha\cos\Omega\sin\theta \tag{11}$$

$$B = -\cos\delta\sin\alpha\sin\Omega \tag{12}$$

$$C = -\sin\delta\cos\alpha\sin\theta - \sin\delta\sin\alpha\cos\Omega\cos\theta \tag{13}$$

The point P has latitude θ and hour angle ϕ, as depicted in Fig. 1. The angles ϕ_{sr} and ϕ_{ss} (expressed in radians in the limits of integration of equation (9)) depend on the latitude θ and the solar declination δ, as is the case for a horizontal surface, and, in addition, on the slope (α) and aspect (Ω) of an insolated surface.

4. THE SUNRISE- AND SUNSET-HOUR ANGLES ON A SLOPING SURFACE

4.1 The Basic Equation

Direct solar radiation is tangential to a surface when the component of the direct solar radiation normal to the surface is zero. This happens when solar radiation first shines upon a surface (sunrise), or when it last shines upon a surface (sunset) on any clear-sky day. This condition is expressed by the following scalar-product statement:

$$\cos\zeta = \boldsymbol{b}\cdot e''_r = 0 \tag{14}$$

The radial unit vector e''_r is given in equation (6). From Fig. 4, the solar-beam vector (**b**) is:

$$\boldsymbol{b} = \tau\ \varepsilon I_0 \left(-\cos\delta\ \boldsymbol{i} - \sin\delta\ \boldsymbol{k}\right) \tag{15}$$

Performing the scalar product in equation (14) leads to the following expression in terms of the hour angle:

$$A\cos\phi + B\sin\phi + C = 0 \tag{16}$$

where A, B, C depend on the slope, aspect, latitude, and solar declination and are given by equations (11), (12), and (13), respectively.

The solutions of equation (16) can be found with iterative algorithms available in commercial numerical packages (Excel, Matlab, Mathematica, for example) or programmed anew [23]. Equation (16) has either two solutions or none. When solutions exist, one is in the interval [0, π], herein denoted by ϕ_{ss}^* , the hour angle for sunset. The other is in the interval [- π , 0] , denoted by ϕ_{sr}^* , the hour angle for sunrise, which is negative due to the convention of this paper to setting the noon hour angle equal to zero. Equation (16) does not have solutions when the latitude, solar declination, aspect, and slope at a site are such that there is 24-hour insolation. The effect of the Earth's near spherical shape shading on slopes needs to be considered before the solution angles ϕ_{sr}^* and ϕ_{sr}^* can be related to the times of sunrise and sunset, respectively, as shown next.

4.1.1 Adjustments needed because of shading of slopes by the Earth's near spherical shape

The solution hour angles ϕ_{sr}^* and ϕ_{ss}^* from equation (16) may or may not equal the actual sunrise-hour or sunset-hour angles, respectively. This is caused by the curved Earth's geometry that shades a slope in variable form as the earth rotates. To validate this assertion, it is helpful to first derive the sunrise- and sunset-hour angles when a surface is horizontal. Equation (16) simplifies to the following expression in the case of a horizontal surface (that is, $\alpha = \Omega = 0$):

$$\cos\phi = -\frac{C}{A} = -\tan\delta\,\tan\theta \tag{17}$$

from which follow the well-known corresponding expressions for the sunrise-hour and sunset-hour angles (in radians) on horizontal terrain:

$$\phi_{sr0} = -\cos^{-1}(-\tan\delta\,\tan\theta) \qquad -\pi \le \phi_{sr0} \le 0 \tag{18}$$

$$\phi_{ss0} = \cos^{-1}(-\tan\delta\,\tan\theta) \qquad 0 \le \phi_{ss0} \le \pi \tag{19}$$

The times of sunrise and sunset associated with equations (18) and (19) are $t_{sr0} = \phi_{sr0} / \omega$ and $t_{ss0} = \phi_{ss0} / \omega$, respectively, in which the rotational angular velocity is approximately $\omega \cong 2\pi$ /24 hr. t_{sr0} is negative, meaning that it precedes solar noon. In general, the duration of daily insolation (in hours) is D = $t_{ss0} - t_{sr0} = (\phi_{sr0} - \phi_{ss0}) / \omega$. In the northern and southern hemispheres, locations with a latitude $90° - |\delta| \le \theta \le 90°$ (with solar declination δ being positive or negative depending on the day of the year according to equation (1)) are insolated 24 hrs daily. In this case, $\phi_{sr0} = -\pi$ and $\phi_{ss0} = \pi$.

The derivation of a rule to determine the sunrise- and sunset-hour angles on a sloping surface is furthered with the aid of Fig. 6, which shows a view of the Earth from over the north pole and perpendicular to the ecliptic plane when $\delta = 23.45°$ (summer solstice). For the sake of argument, Fig. 6 depicts a downward slope starting at point P. The slope has aspect $\Omega = -90°$ (due east). If the surface at point P were horizontal then the sunset-hour angle there would be ϕ_{ss0} given by equation (19). In Fig. 6 the theoretical sunset-hour angle ϕ^*_{ss} obtained from equation (16) is smaller than ϕ_{ss0} (given by equation (19)). In this instance, the actual sunset-hour angle equals ϕ^*_{ss} , that is, $\phi_{ss} = \phi^*_{ss}$, because at angle ϕ_{ss0} the sloping surface is shaded by the curved shape of the Earth.

Fig. 7 shows the continuation of the situation introduced in Fig. 6, now with point P emerging from darkness. The theoretical sunrise-hour angle ϕ^*_{sr} from equation (16) is smaller (that is, more negative) than ϕ_{sr0} (from equation (18)). In this instance, the theoretical sunrise-hour angle does not equal the actual sunrise-hour angle. Rather, the actual sunrise-hour angle equals ϕ_{sr0} , the sunrise-hour angle at P on a horizontal surface. This is so because at angle ϕ^*_{sr} the curved Earth's geometry shades point P.

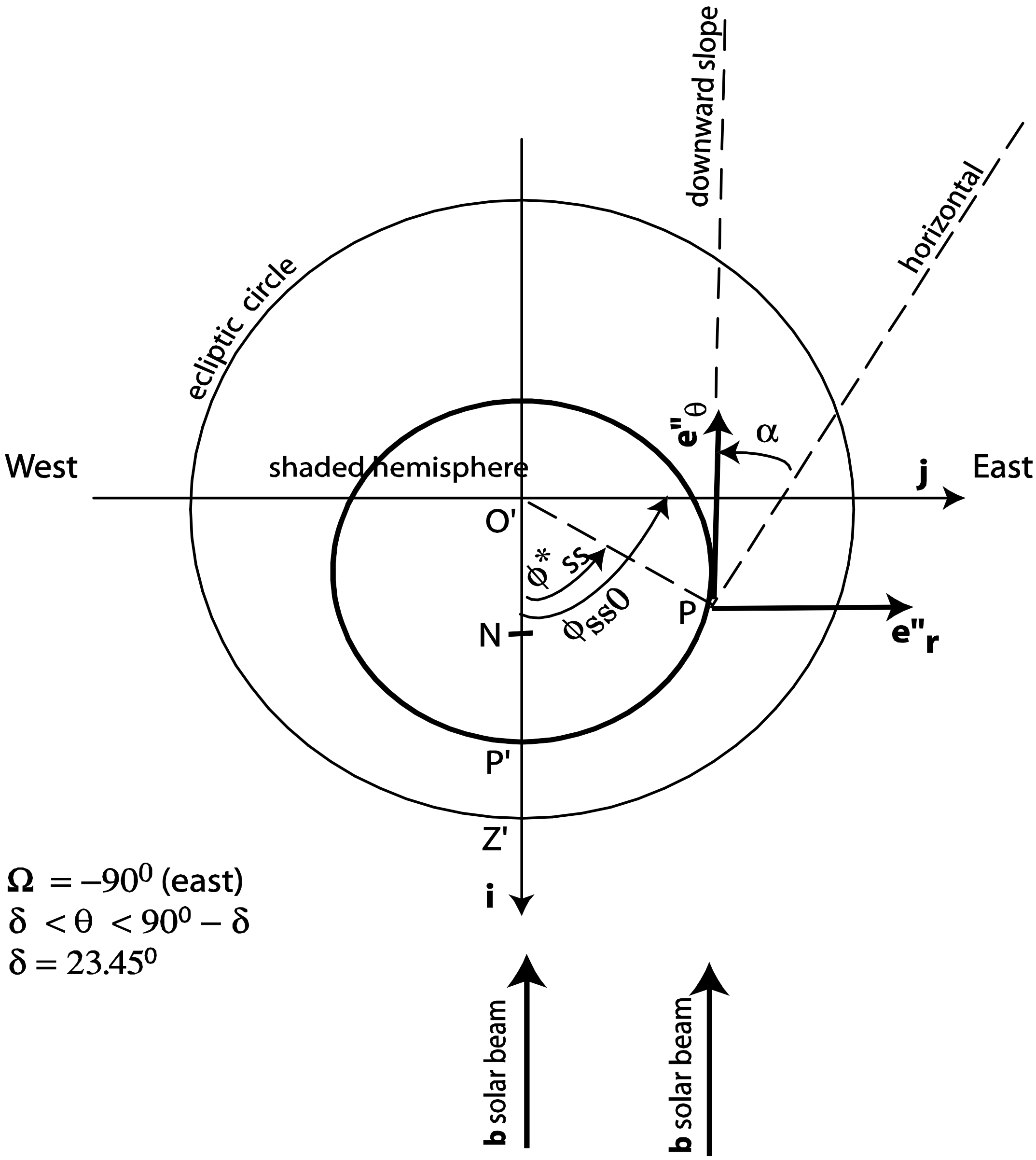

Fig. 6. View from over the north pole and perpendicular to the ecliptic plane showing a slope and aspect that reduce the sunset-hour angle relative to that associated with a horizontal surface (ϕ_{ss0}). The actual sunset-hour angle equals the theoretical angle (ϕ^*_{ss}) in this case

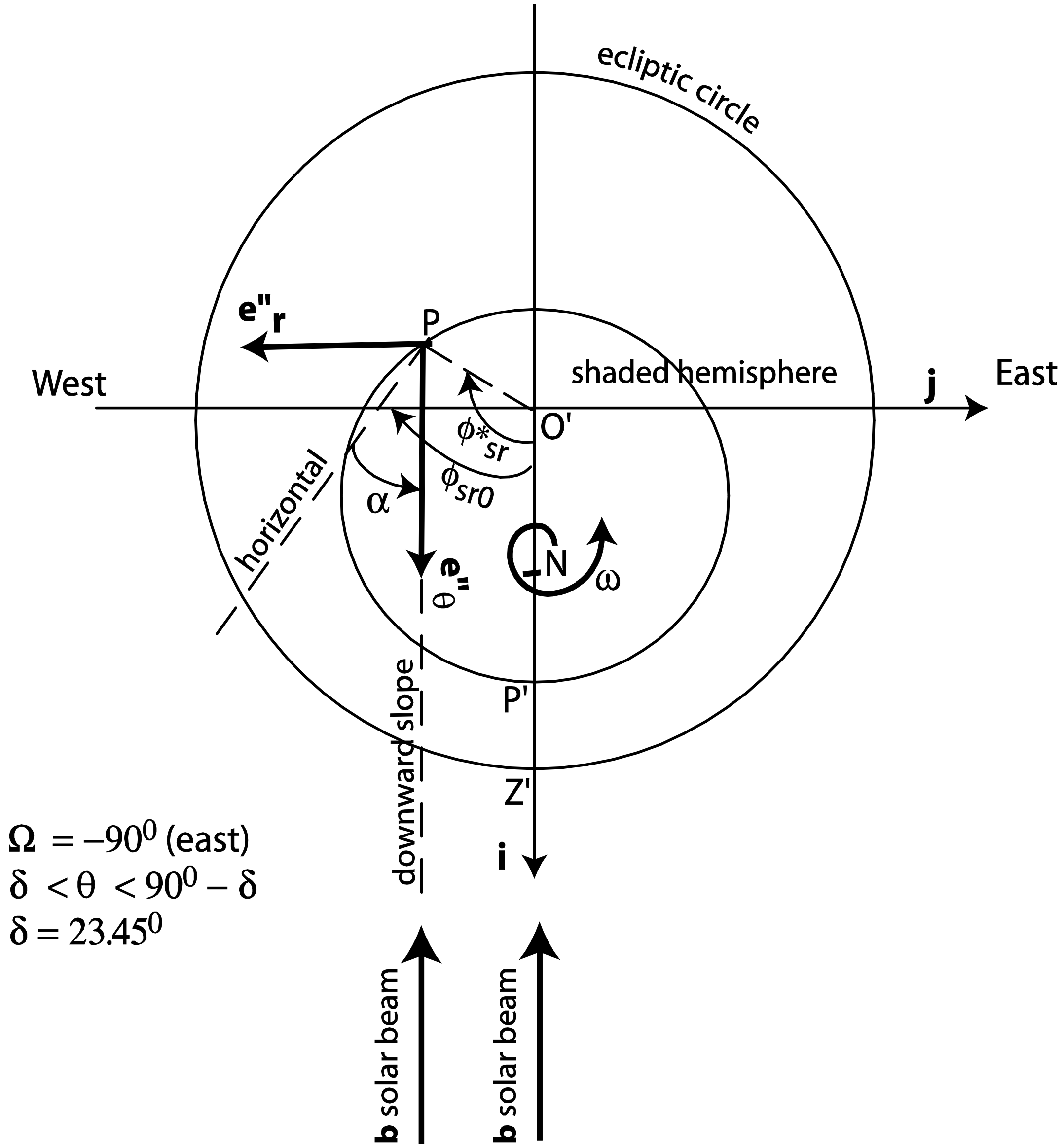

Fig. 7. View from over the north pole and perpendicular to the ecliptic plane showing a slope and aspect for which there is shading of the sloping surface at the theoretical sunrise-hour angle (ϕ^*_{sr}). The actual sunrise-hour angle equals ϕ_{sr0} in this case.

The implication from Figs. 6 and 7 is that the solutions ϕ^*_{sr} and ϕ^*_{ss} of equation (16) equal the actual sunrise- and sunset-hour angles, respectively, only when the sloping surface is not shaded at ϕ^*_{sr} or at ϕ^*_{ss}. This same conclusion can be arrived at by analyzing surfaces of arbitrary declination, slope and aspect.

4.1.2 The decision rule

The preceding arguments lead to the following rule for determining the actual sunrise- and sunset-hour angles, ϕ_{sr} and ϕ_{ss} (in radians), respectively, in terrain of arbitrary slope, aspect, for any latitude and solar declination:

$$\phi_{sr} = \textbf{\textit{the larger of}} \left(\phi_{sr}^{*}, \phi_{sr0}\right) \qquad -\pi \le \phi_{sr} \le 0 \qquad (20)$$

$$\phi_{ss} = \textbf{\textit{the smaller of}} \left(\phi_{ss}^{*}, \phi_{ss0}\right) \qquad 0 \le \phi_{ss} \le \pi \qquad (21)$$

In which ϕ_{sr}^{*} and ϕ_{ss}^{*} are the solutions to equation (16), ϕ_{sr0} and ϕ_{ss0} are obtained from equations (18) and (19), respectively.

Some combinations of latitude, solar declination, slope, and aspect produce 24-hr daily insolation. In this case, $\phi_{sr} = -\pi$ and $\phi_{ss} = \pi$. Once ϕ_{sr} and ϕ_{ss} are determined, they can be used in equation (9) to calculate the daily direct solar radiation input provided that the atmospheric transmissivity (τ) is known.

The duration of daily insolation (in hours) is given by the following expression:

$$D = t_{ss} - t_{sr} = \frac{\phi_{ss}}{\omega} - \frac{\phi_{sr}}{\omega} \qquad (22)$$

in which t_{sr} and t_{ss} are the times of sunrise and sunset, respectively, and ω is the rotational speed of the earth (about 2 π radians / 24 hr).

Fig. 8 shows calculation of the duration of daily insolation for clear-sky conditions in north facing (Ω = 0 with downward slope), west facing (Ω = 90° with downward slope), and south facing (Ω = 180° with downward slope) sloping terrain, for a northern latitude of θ = 23.45° and solar declination δ = 23.45° (summer's solstice). The graphs for west-facing and south-facing terrain show that the duration of insolation decreases with increasing slope. In the south-facing case, the duration of insolation vanishes when the slope of south-facing terrain approaches α = 90° (vertical slope). Arbitrary combinations of Ω, θ, and δ could be entertained equally as easily.

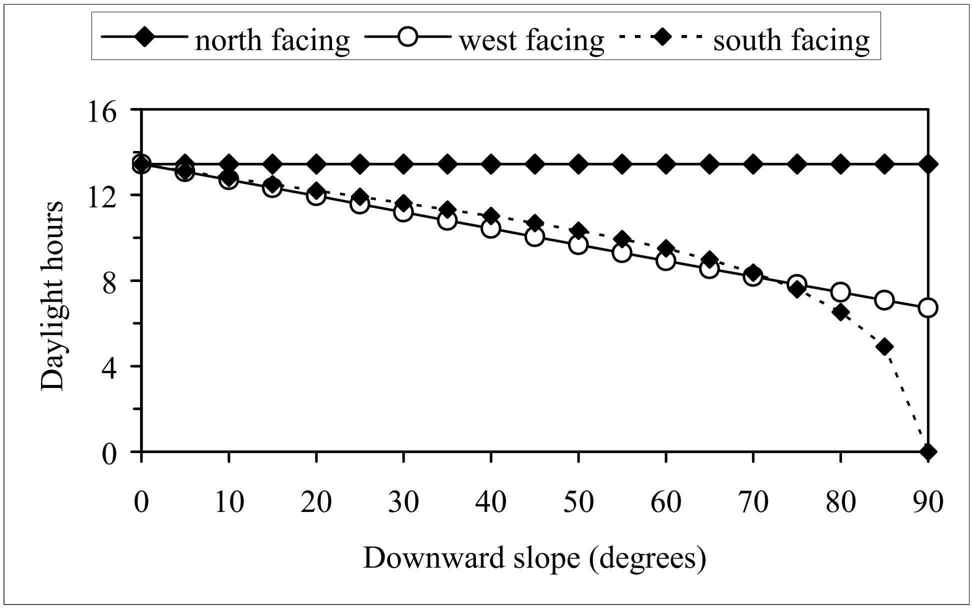

Fig. 8. Daylight hours, or duration of daily solar insolation with clear-sky conditions, for a terrain of aspect Ω = 0 (north-facing slope), Ω = 90° (west-facing slope), and Ω =180° (south-facing slope), as a function of slope (downward or positive in this instance). The latitude of the sloping terrain in this example is θ = 23.45°, with declination δ =23.45° (summer solstice in the northern hemisphere)

5. DOUBLE SUNRISE AND SUNSET

There are northern and southern high latitudes (high in absolute value in the latter case) that, when combined with steep slopes, produce two sunrises (hour angles ϕ_{sr1}, ϕ_{sr2}, with $\phi_{sr2} > \phi_{sr1}$) and two sunsets (hour angles ϕ_{ss1}, ϕ_{ss2}, with $\phi_{ss2} > \phi_{ss1}$). This situation occurs when, for example, the critical slope α -shown in Fig. 4- is exceeded, producing in this instance a shaded slope during an interval that would otherwise (i.e., without the slope) be lighted. Yet, that slope may be insolated prior to and after the interval of darkness. This situation calls for two sunrises and two sunsets. The first sunrise (ϕ_{sr1}) occurs when the sun first shines on the slope on any clear-sky day. The first sunset (ϕ_{ss1}) ends the first period of insolation, at which time darkness sets in on the slope until the second sunrise (ϕ_{sr2}) reinitiates insolation. The latter ends with the second sunset (ϕ_{ss2}). In this case the energy-input equation (9) must be rewritten as follows:

$$I_S = \int_{\phi_{sr1}}^{\phi_{ss1}} \boldsymbol{b} \cdot e''_r \; d\phi + \int_{\phi_{sr2}}^{\phi_{ss2}} \boldsymbol{b} \cdot e''_r \; d\phi \quad (23)$$

in which all the intervening terms in the integrands are exactly as defined in association with equation (9), and the solar-beam vector **b** is specified in equation (15). The angles that appear as limits of integration in equation (23) are expressed in radians. In equation (23), $\phi_{sr1} = \phi_{sr0}$, $\phi_{ss1} = \phi^*_{sr}$, $\phi_{sr2} = \phi^*_{ss}$, and $\phi_{ss2} = \phi_{ss0}$, where ϕ_{sr0} and ϕ_{ss0} were defined in equations (18) and (19), respectively, and correspond to the horizontal-case hour angles; ϕ^*_{sr}, ϕ^*_{ss} are the solutions to equation (16). The duration of daily insolation when there are two times of sunrise and two times of sunset is:

$$D = \frac{1}{\omega}\left(\phi_{ss1} - \phi_{sr1} + \phi_{ss2} - \phi_{sr2}\right) \tag{24}$$

where ω = 2π radians / 24 hr.

To illustrate the occurrence of double sunrise and sunset –as well as the calculation of the various hour angles introduced above- let the slope (α), latitude (θ), solar declination (δ), and aspect (Ω) be $\pi/3, \pi/3$, 23.45, and 0 degrees, respectively. In this case, ϕ_{sr1} = ϕ_{sr0} $\cong -2.421$, ϕ_{ss1} = ϕ^*_{sr} $\cong -0.721$, ϕ_{sr2} = ϕ^*_{ss} $\cong$ 0.721, and $\emptyset_{ss2} = \emptyset_{ss0} \cong 2.421$. Notice that the sloping surface is dark when the hour angle is zero in this instance. For the sake of contrast, let the aspect Ω be nonzero, say, equal to $\pi/4$, and using the same α, θ, δ as before, yields: ϕ_{sr1} = ϕ_{sr0} $\cong -2.421$, ϕ_{ss1} = ϕ^*_{sr} $\cong -2.216$, ϕ_{sr2} = ϕ^*_{ss} $\cong -0.0669$, and ϕ_{ss2} = ϕ_{ss0} $\cong$ 2.421. Evidently, in this second example the sloping surface is insolated when the hour angle is zero. Any other combination of the variables controlling the duration of daily insolation can be handled similarly with the equations developed in this paper.

The method developed in this paper to calculate the duration of daily insolation follows directly from equations (16), (18)-(19) and (22) or (24). It avoids the use of equivalent slopes (see, [14]), in which an actual sloping surface at a given latitude and longitude is replaced by an (equivalent) horizontal surface placed at a different longitude and latitude for the purpose of calculating the times of sunrise and sunset on the actual slope. This subterfuge introduces unnecessary complications in the calculation of energy input by direct solar radiation, which are avoided by the direct method of calculation of this work.

6. CONCLUSION

A closed-form equation for the duration of daily insolation on surfaces of arbitrary slope and aspect has been derived in this article. The key to obtaining the duration of daily insolation lies on finding the roots of the trigonometric equation A$\cos\phi$ + B$\sin\phi$ + C = 0 for the hour angle ϕ, in which the coefficients A, B, C encompass the geometric factors that govern the flux of solar radiation normal to a surface, namely, slope, aspect, solar declination, and latitude. The method to calculate the duration of daily insolation developed in this paper can be efficiently implemented for use in a variety of agricultural meteorologic, hydrologic, ecologic, and climatic applications, a key one being the calculation of clear-sky daily solar radiation input to sloping surfaces.

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COMPETING INTERESTS

Author has declared that no competing interests exist.

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