British Journal of Environment & Climate Change
3(3): 480-498, 2013

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Potential and Merits of the Eigenvalues Method to Simulate the Conjunctive Use of Groundwater and Surface Water

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Author's contribution

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Research Article

Received 1st November 2012
Accepted 28th June 2013
Published 15th September 2013

ABSTRACT

It is necessary predict the effect of aquifer stresses in surface water and wetlands and consider the mutual effects that are produced by the conjunctive use of surface water and groundwater. This was originally made with very simple idealized analytical methods. The next development was the application of finite differences or finite elements numerical models, but poses problems when the model has to be run many times to analyze different management alternatives. When aquifer behavior is linear, as in confined, semi-confined, or unconfined aquifers with not too large changes in its saturated thickness, it is possible to apply the superposition strategy through influence functions. That has simplified significantly modeling and improved the effectiveness of management models. However, for large models, long modeling periods and a large number of alternatives, it is needed to handle and store many influence functions and to consider and store all the previous stresses. In that case, the eigenvalue method can be a more appropriated option. This approach solves the spatially discretized flow equation explicitly and continuously in time, obtaining modal orthogonal components through very simple explicit state equations in function of time. To reduce the computational load, the simulation can be simplified with appropriate truncation using only dominant modes of the components at the expense of a small error. Efficient methods have been developed to get the modal components as well as to perform truncation with limited errors.

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Keywords: Groundwater flow; linearization; numerical models; eigenvalue method; truncation; conjunctive use.

1. INTRODUCTION

The interest in conjunctive use of groundwater and surface water is increasing because of the growing demand for water supply, irrigation, and the environment in wetlands and riparian habitats, and the effect of declining water resources due to global warming. The characteristics and behavior of surface waters and aquifers are different and complementary to fill needs of water quantity and quality. Aquifers can provide storage and distribution that can be combined with surface water and hydraulic structures to increase water availability timely and at lower cost than if both resources are used separately. Aquifers store large volumes of water, tens to hundreds of times their annual recharge. Also the storage provided by a relatively small fluctuation of groundwater in unconfined aquifers considerably exceeds the available or economically feasible surface storage. That allows the use of water in storage during dry seasons or droughts and the use of the subsurface space for storing surface or reclaimed water.

In river-aquifer systems groundwater pumped initially comes from aquifer storage. With time the cone of depression extends, storage is depleted more slowly and increases the capture from surface water. The capture comes in the form of increase from recharging boundaries and from decrease in springs, wetlands or river flow. The capture can be expressed as an instantaneous flow divided by the flow rate which eventually becomes asymptotically equal to one in infinite time. As the distance from the river to the well increases, also the time required reaching a determined proportion of capture increases, and so does the aquifer storage depletion. The dynamics of ground water is qualitatively and quantitatively different from surface water. Groundwater behaves almost deterministically and its flow is much slower than the flow of rivers, whose behavior is stochastic and by nature unpredictable. So aquifers can be used more intensively in dry seasons or in droughts and surface water stored on dams or derived from rivers should be used more in wet seasons. This strategy is what we call *alternate conjunctive use* to distinguish from the artificial recharge. It is widely done spontaneously by the farmers in the Mediterranean basins of Spain [1]. *Artificial recharge* and *alternate conjunctive use* are used to store surface water in aquifers.

The solution of the groundwater flow equation can be obtained with numerical and analytical methods. This was originally made with analytical methods that represented natural systems in a highly idealized manner. Up to now only a few analytical solutions for limited aquifers have been developed. The analytical solution [2] for the depletion of a river fully penetrating a semi-infinite aquifer was one of the first quantitative methods applied to quantify capture through the concept of stream depletion factor (*sdf*) introduced by Jenkins [3]. That value is the time in days at which accumulated change in stream flow volume equals 28% of the volume pumped by a well pumping at a constant rate. The lines of equal *sdf* obtained from the modeling results of a finite aquifer were mapped to be used in a management model [4]. The artifice of using the *sdf* concept in finite aquifers has been used until very recently and probably will continue to be widely used [5,6] because of its simplicity. Numerical models are much more flexible and can be applied to aquifers with complex geometries and heterogeneous properties. They are not only very useful and robust, but essential tools for the analysis of aquifer response.

To simulate the operation of a conjunctive use system it is necessary to include the storage in dams, the flow through canals, pipelines and rivers, diversions or uptakes for the different uses, the aquifer behavior and the flow interchanges between aquifers and rivers. Surface and subsurface components must be simultaneously simulated, owing to their hydraulic interactions, the operating rules and legal, institutional and social constraints inherent in surface and subsurface components. Besides if we want to take into account the stochastic behavior of surface runoff and its uncertainties, aggravated by climate change and the increase of water demand, it would be desirable to run repeated simulations of many alternatives over long time periods. That implies large and repeated simulations of the aquifer of the system. But, only the results of hydraulic head in a few points, and the aquifer river relationship, are needed from aquifer simulation [7].

Capture that results in a loss of water in springs, rivers, and wetlands is extremely important because of depletion of groundwater dependent ecosystems and reduction of surface water supplies with water rights. The simulation of multiple captures in different areas and reaches of the river throughout prolonged periods of time also requires large and repeated simulations of a model [8].

The influence function concept, widely used in physics and engineering, adds new improvements and possibilities. The utilization of functions of influence needs the linearity of the equation of flow, as does the eigenvalue method. Later, several aspects of linearity are discussed in some detail. Also, there is discussed the concept of linear reservoir and the structure of its simple solution that is identical to that of the eigenvalue method. Then, the eigenvalue method and its applications are presented.

2 LINEAR MODELS

2.1 Linearity of the Flow Equation

The flow equation for confined aquifers or for aquifers where transmissivity, or the product of hydraulic conductivity multiplied for the saturated depth, does not significantly change during the exploitation, is the well-known partial derivatives equation in 2D

$$\frac{\partial}{\partial x}\left(T_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(T_y\frac{\partial h}{\partial y}\right)+Q(x,y)=S\frac{\partial h}{\partial t} \quad (1)$$

Where h is hydraulic head, T_x and T_y are the principal components of transmissivity tensor, S is storage coefficient, $Q(x,y)=Q_d(x,y)+\sum_i Q_i\,\delta(x-x_i,y-y_i)$ is composed by distributed and point stresses and δ( ) is the Dirac delta function. In the flow equation the sign of the stress is positive for aquifer recharge and negative for extraction.The solution needs adequate initial h_0 (x,y,0) and boundary conditions (BC). BC can be specified flow or no flow (Newman), specified or constant head (Dirichlet), or head dependent flow (Cauchy). Equation (1) can be written in the following way

$$\mathfrak{L}(\mathrm{h})+Q(x,y)=S(x,y)\frac{\partial h}{\partial t} \quad (2)$$

Where $\mathfrak{L}$ is a lineal operator

$$\mathfrak{L}()=\frac{\partial}{\partial x}(T_x\frac{\partial}{\partial x})+\frac{\partial}{\partial y}(T_y\frac{\partial}{\partial y})$$

The solution of equation (1) or (2) is the sum of three components,

$$h(x,y,t) = h_s(x,y) + w(x,y,t) + \sum s_m(x,y,t)$$

a) A steady state solution $h_s(x,y)$, with the same boundary conditions of (1), Newman, Dirichlet and Cauchy, and zero stress.

$$\mathfrak{L}[h_s(x,y)] = 0 \tag{2a}$$

b) A transient solution w(x,y,t) for zero stress, with zero value in the three types of boundary conditions. The initial condition is the difference of the initial condition of the whole problem $h_0(x,y,0)$ minus the steady state solution $h_0(x,y,0) - h_s(x,y)$ and

$$\mathfrak{L}[w(x,y,t)] = S(x,y)\frac{\partial w}{\partial t} \tag{2b}$$

c) The sum of all transient solutions s_m with zero initial and boundary conditions for each distributed and point stresses

$$\mathfrak{L}[s_m(x,y,t)] + Q(x,y) = S(x,y)\frac{\partial s_m}{\partial t} \tag{2c}$$

being $Q(x,y)$ any linear combination of the distributed or point stresses.

This decomposition is schematized in the Fig. 1. Three steps are usually followed to obtain the influence produced by an additional pumping or recharge on aquifer heads or in flow capture from a river reach. The first step is a simulation without the added stress. The second step is to re-run the simulation with no other changes except for the added stress. The third step is to compute changes from the base case for selected simulation times [8]. To compute the influence on heads or flows at any point in the aquifer, in the case of linear models it is clear from the above described decomposition in steady and transient components that it is not needed run two times the model, but only one with zero initial and boundary conditions. The capture produced by pumping in an aquifer with linear behavior is identical to that which would occur in it, with null initial and boundary conditions. Likewise, as it is correctly pointed in [8], and because in an aquifer without recharge and zero initial and boundary conditions, there are no flow or stream lines, does not exist up gradient nor down gradient, and does not exist any base flow originated by any recharge, statements like the following: capture depends on rates and directions of groundwater flow, capture mostly occurs in stream reaches down gradient of pumping locations, or capture is limited to the fraction of base flow that originates in the pumped area, are incorrect.

Instead of running the model for each alternative of exploitation, the response function technique has been proposed [9,10,11]. If the simulation is divided in periods of the same Δt length, each period ending in time nΔt can be identified as the period n. If h^i_{nat} is the head in point *i* due to unmodified conditions and h^i_{inf} is its head influenced by the stresses, the effect of K different stresses, of different size at every period, at the end of the n period is given by:

$$h_{inf}^{i}(n\Delta t) = h_{nat}^{i}(n\Delta t) + \sum_{k=1}^{K}\sum_{m=1}^{n} P_{k(m\Delta t)}\, \delta_k^i[(n-m+1)]\Delta t \quad (3)$$

Where $P_k(m\Delta t)$ is the magnitude of stress k and $\delta_k^i(n\Delta t)$ is the influence of a k unit stress on h^i head. Very often the most interesting effect to control is the capture of flow in a river reach. Calling Q_{inf}^j(nΔt) and Q_{nat}^j(nΔt) the influenced or natural flow, respectively, and δ_k^j the influence of a k unit stress on the flow of reach j, it results:

$$Q_{inf}^{j}(n\Delta t) = Q_{nat}^{j}(n\Delta t) + \sum_{k=1}^{K}\sum_{m=1}^{n} P_k(m\Delta t)\delta_k^j[\,(n-m+1)n\Delta t] \quad (4)$$

In aquifers of medium size it is common to use influence functions for monthly periods for 20 to 50 years. Stresses do not need to be limited to point stresses; they can be point, distributed or a combination of different kinds of pumping or recharge. In any case, influence functions, δ_k^j or δ_k^i, where flow or head is required, must be stored, and similarly all P_k stresses during all the previous periods. This requires in most cases an important computation and storage capacity, although usually much less that in the case of the full run of the aquifer model for every alternative. In case we are only interested in capture or in heads in a few points in the aquifer, *simulation by superposition* is clearly superior to the *complete simulation* [12,13]. Usually stresses on aquifers can be decomposed into a relatively small number of unitary stresses; influence functions can be obtained to simulate the aquifer. This may provide significant advantages in many management problems.

In [8] is presented a method to simulate and map capture using an automated procedure to run the model repeatedly, each time with a well in a different location. To make the capture map of the San Pedro aquifer, Arizona, cells were considered at every fourth row and every fourth column, requiring 1530 model runs to compute capture values on a grid with a spacing of 1 km in both horizontal dimensions. In the model of the upper Deschutes Basin, Oregon, pumping locations included all of the active cells in the model domain, requiring 53,589 simulations. As for the San Pedro model, the Deschutes model was modified to start with predevelopment steady state conditions and to simulate 100 years with 100 1-year time steps and constant stresses. An improvement to the use of the influence functions is the implementation of the method of the eigenvalues. This method solves the differential equation of unsteady flow of aquifers explicitly and continuously over time. This solution has significant advantages when running the model for several alternatives during multiple time periods. This advantage comes in conjunctive use problems, especially when the system has several aquifers, canals and pipelines, reservoirs, and many areas of water demand. The eigenvalues method can also greatly simplify the mapping of the capture in different river reaches, of a unitary pumping along the aquifer.

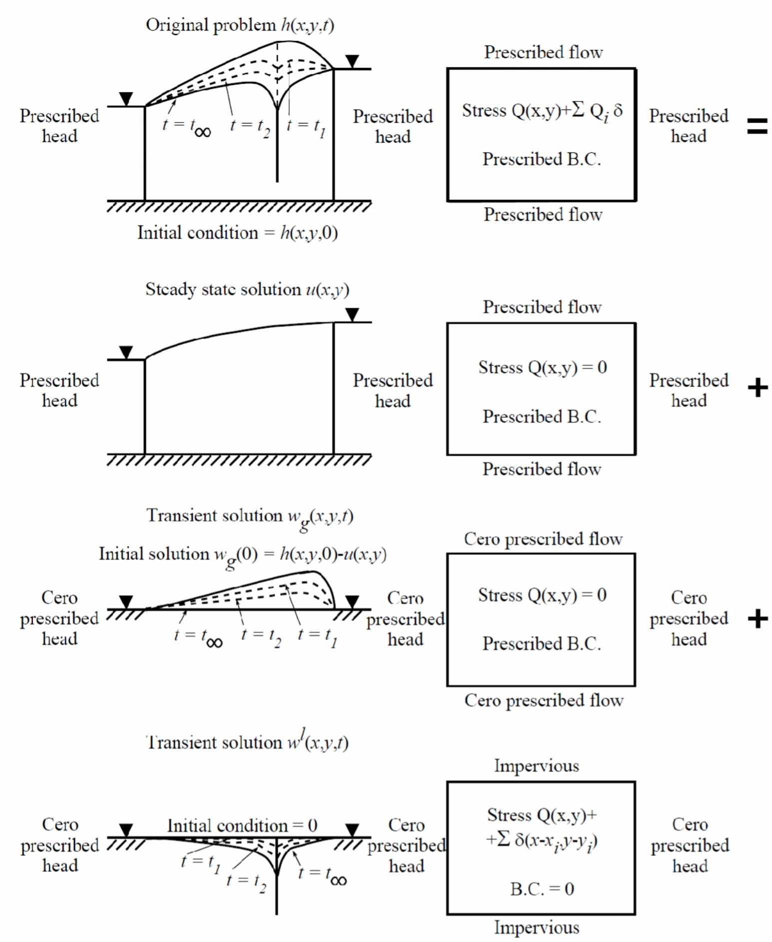

Fig. 1. Components of the piezometríc head, initial conditions, boundary conditions, and aquifer stresses

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Before describing the method it seems appropriate to consider both the linear reservoir model and the so-called embedded multi-reservoir model, the last being based on the eigenvalue solutions.

2.2 The Linear Reservoir Model

The model represents the aquifer as a single cell with a stored volume of water, V, receiving recharge and subject to pumping. The natural outlets of the aquifer are rivers or springs. The volume stored above the outputs is V. Discharge is D = αV, being α coefficient of discharge with dimensions of $[T^{-1}]$. If R is aquifer recharge per unit time, the differential equation describing the balance in the cell is:

$$R - \alpha V = \frac{dV}{dt} \tag{5}$$

Its solution for constant R is:

$$V = V_0 e^{-\alpha t} + \frac{R}{\alpha}(1 - e^{-\alpha t}) \quad (6a)$$

Or expressed in terms of discharge:

$$D = D_0 e^{-\alpha t} + \text{R}\,(1 - e^{-\alpha t}) \quad (6b)$$

The unicellular model is simple and easy to include in any calculation scheme. As the model is linear, superposition can be applied. The effect of a continuous pumping B considering zero volume and flow initial conditions results in the last terms of equations (6a) and (6b) to be changed respectively by

$$\frac{B}{\alpha}(1 - e^{-\alpha t}) \text{ and } \text{B}(1 - e^{-\alpha t}). \quad (7)$$

2.3 The Eigenvalue Solution of the Linear Flow Equation

If we know the solution of the steady state $h_s(x, y)$ of equation (2a) with the same boundary conditions of (1), we only need to solve for zero B.C the equation

$$\mathcal{L}[w(x,y,t)] + Q(x,y) = S(x,y)\frac{\partial w(x,y,t)}{\partial t} \quad (8)$$

Using the method of separation of variables

$$w(x,y,t) = A(x,y)l(t) \quad (9)$$

Yields the following two equations [17]:

$$\mathcal{L}[A(x,y)] + \alpha S(x,y)A(x,y) = 0 \quad (10)$$

$$\partial l(t)/\partial t + \alpha\, l(t) = 0 \quad (11)$$

Equation (10) is subject to the same zero boundary conditions as w in equations (2b) and (2c). It is a Sturm-Liouville problem which implies that it has only a solution for some values of α [14, 15]. Each of them is parallel with their eigenfunction $A_i(x, y)$, resulting in infinite eigenvalues α_i; all of them are real and positive. For each pair eigenvalue-eigenfunction the solution is given by:

$$w(x,y,t) = \sum_1^\infty A_i(x,y)\left[\frac{l_i(0)e^{-\alpha_i t}}{\alpha_i} + \frac{1-e^{-\alpha_i t}}{\alpha_i}\,\square\, Q(x,y)A_i(x,y)d\Omega\right] \quad (12)$$

The eigenfunctions form an orthonormal basis with respect the storage coefficient S (x, y)

$$\square\, A_i(x,y)A_j(x,y)S(x,y)dxdy = \delta_{i,j} \quad \text{i.e. =1 If i=j, or =0 if i≠j.} \quad (13)$$

It is more convenient and efficient to express the solution at the base of the eigenvectors $l_i(t)$ obtained explicitly and so any state of the system is defined by the values of l_i.

$$l_i(\mathrm{t}) = \left[\frac{l_i(0)e^{-\alpha_i t}}{\alpha_i} + \frac{1-e^{-\alpha_i t}}{\alpha_i} \square\, Q(x,y)A_i(x,y)d\Omega\right] \quad (14)$$

The initial conditions are reflected in the values of $l_i(0)$ obtained as:

$$l_i(0) = \square\, h(x,y,0)S(x,y)A_i(x,y)dxdy \quad (15)$$

An important practical aspect is the determination of river flows and aquifer volumes that correspond to each component of the ith eigenvalue. Let's call F_i to the volume under the surface of the eigenfunction A_i.

$$F_i = \iint S(x,y)A_i(x,y)dxdy \quad (16)$$

The volume under each component is l_iF_i and the total volume above the zero level is

$$\mathrm{V}(\mathrm{t}) = \sum_1^\infty l_i F_i \quad (17)$$

It is easy to show that the action $Q = \iint Q(x,y)dxdy$ is partitioned between each component as b_iQ, being

$$b_i = \frac{\square\, Q(x,y)A_i(x,y)dxdy}{Q} \quad \text{being} \quad \sum b_i = 1 \quad (18)$$

Comparing the results with the discharge of a linear reservoir it appears that the discharge from the aquifer through their boundaries is equivalent to the output from infinite linear deposits with a discharge coefficient α_i. The external action Q is split between them in the form biQ. This is the conceptual approach from the embedded multicellular models expressed schematically in Fig. 2. Very few analytical solutions of this problem have been derived and those existing are in general for 1D infinite aquifers, with simple external actions and initial conditions. With the proposed methodology, the analytical solutions for more complex cases have been obtained, including any external action and any initial condition, [14].

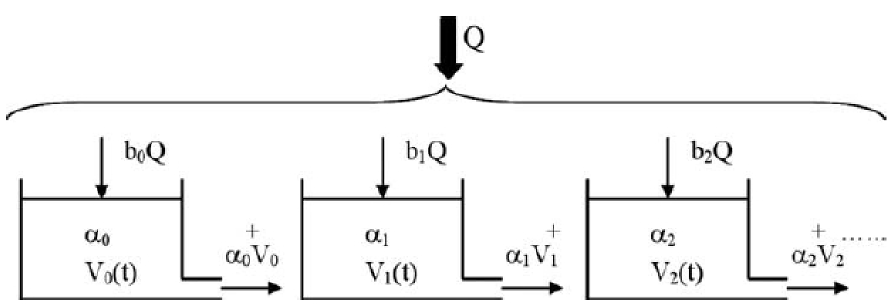

Fig. 2. Conceptualization of the embedded multicelular model

2.3.1 Embedded multicellular model

Here some aspects of the two simplest solutions are discussed. Let's consider a rectangular aquifer of length L and width W in the directions x and y connected to a river at the side x = L with two possible situations: perfect connection or partial connection.

2.3.1.1 Perfectly connected river

The eigenfunction are

$$A_{m,n}(x,y) = \sqrt{\frac{2}{SLW}}\, cos\frac{(2m+1)\pi}{2L}\, x \qquad \text{For m = 0,1, 2, . ; n= 0} \qquad (19a)$$

$$A_{m,n}(x,y) = \sqrt{\frac{2}{SLW}}\, cos\frac{(2m+1)\pi}{2L} x\, cos\,\frac{n\pi}{W}\, y \qquad \text{for m = 0,1, 2, . ; n= 1, 2 ,3 .} \qquad (19b)$$

And the eigenvalues $\alpha_{n,n} = \frac{(2m+1)^2\,\pi^2 T}{4SL^2} + \frac{n^2\pi^2 T}{S\,W^2}$

The b values for actions uniformly distributed in the aquifer are

$$b_{m,n} = \frac{8}{\pi^2(1+2m)^2} \qquad \text{for n = 0} \qquad (19c)$$

And is zero for n ≠ 0, i.e. when there are components the directions parallel to the river OY. Two important consequences are:

1) Any pumping at the same distance from the river will produce the same detraction, although differently distributed along their course.

2) Fi, the volume of water stored below their corresponding eigenfunction is zero being A(x,y) an odd function. Those eigenvalues are valid for the determination of the piezometric heights, but do not participate in the capture of water from the river due to pumping. In heterogeneous aquifers with more complex geometries there are a limited number of modes that explain most of the capture and there is other much more numerous that can be omitted to determine it without incurring major errors.

The eigenvalues of the main modes, without component in the direction 0Y are $\alpha_i = \frac{(2i+1)^2\,\pi^2 T}{4SL^2}$ or $\alpha_i = (2i+1)^2$ α, being α= $\frac{\pi^2 T}{4SL^2}$ the lowest eigenvalue of the most dominant mode. The larger the dimensions of the aquifer (L) and the lower its diffusivity T/S the smaller is α, which implies larger inertia of the aquifer.

The successive values of $b_i = \frac{8}{\pi^2(1+2i)^2}$, from i = 0 onwards, are 0.8106, 0.0900, 0.03242, 0.0165, 0.0100, .. . This shows that in any case using a small number of modes sufficient accuracy can be obtained. In simulations with Δt of one month, in most cases, except for wells very near- the river, five terms suffice to decrease the relative error to less than 0.01.

To preserve the mass balance is advisable to change the last b_i by $b_p = 1 - \sum_0^{p-1} b_i$ [16, 17].

2.3.1.2 River with partial connection

In the case of a river with partial connection, the strength factor of the semi-pervious layer $S_{pr} = \frac{\kappa B}{e}$ must be considered. It depends on, its thickness e, its hydraulic conductivity κ, and their saturated thickness B. The eigenfunction have also separate components in directions OX and OY, being the OY components the same that for the case of perfectly connected river. The partition coefficients and discharge depend on a dimensionless hydraulic connection parameter λ through a factor ρ whose values are the infinite solutions of the equation

$$\frac{\pi\rho}{2}\tan\left(\frac{\pi\rho}{2}\right) = \lambda \qquad \text{Being} \quad \lambda = \frac{S_{pr}L}{T} \tag{20}$$

For the eigenfunction with component on the OY axis the b_i values are zero indeed. For the dominant modes, those with only OX components

$$b_i = \frac{8sin^2(\rho_i\pi)}{\rho_i{}^2\pi^2 + \rho_i \sin(\rho_i\pi)} \tag{21}$$

The eigenvalues are $\alpha_i = \rho_i^2\alpha$; where α is as before. In Table 1 the influence of the resistance of the semipervious layer through the parameter λ can be assessed. Infinite λ is equivalent to the perfect connection and decreases with impaired connection. In these cases, the minor eigenvalue decreases smoothly from λ = 1000 to λ = 10, and becomes 0.3α for λ=1 and 0.04α for λ= 0.1, values of λ that are possible in real situations. For such low values, the volume stored in the aquifer, and as consequence their heads, would be much higher.

Table 1. Values of ρ_i^2as a function ofλ

λ	i=0	i=1	i=2	i=3	i=4
0	0.0004	4.0008	16.0008	36.0008	64.0008
0.01	0.004	4.0081	16.0081	36.0081	64.0081
0.1	0.0392	4.0806	16.0809	36.081	64.081
1	0.3	4.756	16.7945	36.8032	64.8064
10	0.8275	7.5139	21.1743	42.168	70.7687
100	0.9803	8.8228	24.5084	48.0382	79.4141
1000	0.998	8.982	24.9501	48.9022	80.8383

The capture will also be much lower as could be computed from (6a and b). In Table 2 the variation of the partition coefficients b_i is shown when the value of λ varies. For λ = 1000 it is virtually identical to the case of perfect connection, 0.8114, increasing to 0.87 for the lower mode for λ = 10 and being 0.9861 for λ = 1 with very low contribution of the modes that follow it. The need to use more modes is less when λ decreases; i.e. when the ratio L/T decreases, and the semipermeable layer resistance increases. In such cases the error of using a single mode is lower; i.e. the one-cell model is more justified. The influence of the parameter λ on the capture of the river flow by pumping from the aquifer is substantial, being its influence on groundwater levels very important.

Table 2. Variation of the partition coefficients b_i when the value of λ varies

λ	i=0	i=1	i=2	i=3	i=4
0.001	1.0000	2.1 x10^{-8}	1.3 x10^{-9}	2.5 x10^{-10}	8 x10^{-11}
0.01	1.0000	2.1 x10^{-6}	1.3 x10^{-7}	2.5 x10^{-8}	8 x10^{-9}
0.1	0.9998	2.1 x10^{-4}	1.3 x10^{-5}	2.5 x10^{-6}	8 x10^{-7}
1	0.9861	0.0124	0.0011	0.0002	7.7 x10^{-5}
10	0.8743	0.0839	0.0236	0.009	0.004
100	0.8185	0.0908	0.0326	0.0165	0.0099
1000	0.8114	0.0902	0.0325	0.0166	0.0100

3. THE EIGENVALUE DISCRETE METHOD

In cases where there is a calibrated numerical model of finite difference or finite element without significant nonlinearities, the discrete eigenvalue method can be used. For doing it, only the transient component of equations (2b) and (2c) is considered. If only space is discretized, but not time, as it is usually done in the case of finite differences or finite elements, a vector equation system is obtained that can be expressed in matrix form as follows

$$|T|\overline{H} + Q = |SF|\frac{\partial \overline{H}}{\partial t} \tag{22}$$

where | T | is a banded symmetric positive definite matrix of size N by N whose values only depend on the geometry of the system, the space discretization, the boundary conditions and the values of transmissivity. $\overline{H}$ is the N component vector of piezometric heads at each node of the space discretization, Q is a N component vector representing net incoming flow at each node, which changes for each time increment, | SF | is a diagonal matrix of N by N size, with the storage of each node being S Δx Δy, when the method of finite differences is employed, and t is time. The solution is analogous to the analytical solution presented before; equation is split into a component in the space and another in time, [7,15].

$$\overline{H} = |A|\overline{L}(t) \tag{23}$$

From (22) and (23) separated equations similar to (10) and (11) are obtained

$$|T||A| = -|SF||A||\alpha| \tag{24}$$

$$\overline{L}(t)=|e^{-\alpha t}|\,\overline{L}_0 \tag{25}$$

Being $|e^{-\alpha t}|$ a diagonal matrix with terms $e^{-\alpha t}$. Expression (24) is an eigenproblem whose solution provides the eigenvalues α, which are the diagonal components of the matrix $|\alpha|$, and their corresponding eigenvectors, which are the columns of | A |. Both matrices of dimensions N by N are the “modes” of equation (24). Piezometric head can be defined indistinctly by vector H or expressed in the orthonormal basis of the eigenvectors as L. Ortho-normality being expressed by the equation

$$\left|A'\right|\left|SF\right|\left|A\right| = \left|I\right| \quad (26)$$

Where $\left|A'\right|$ is the transpose of matrix $|A|$, and $|I|$ is the identity matrix. The diagonal matrix of eigenvalues $|\alpha|$ is ordered from the lowest value to the greatest. The eigenvectors in $|A|$ are ordered as their companion eigenvalues, the first one corresponds to the smallest and all its components are always positive. The solution of (22) can be obtained from vector L for piezometric heads through (23). But the vector L also contains the complete solution of the state of the aquifer and is much more interesting and easy to operate with very small computational requirements.

When the boundary conditions are not null, vector $|\bar{H}_s|$ must be added to the solution of the piezometric heads. That vector would be obtained for nonzero boundary conditions without any pumping or recharge in the aquifer. This is the solution of equation (2a) as stated above. But this is not necessary when superposition is used, or to determine exclusively the effect of pumping or recharge on river flow, or piezometric heads.

If we use the simulation of the aquifer as a component of a management model, time can be divided into periods of equal length Δt. L_1 can be obtained from L_0 and Q_1, L_2 from L_1 and Q_2, and so on. Thus:

$$\bar{L}_k = |E(\Delta t)|\bar{L}_{k-1} + \left||I| - |E(\Delta t)|\right||\alpha^{-1}|\left|A'\right|\bar{Q} \quad (27a)$$

or $$\bar{L}_k = |E(\Delta t)|\bar{L}_{k-1} + |\Psi||\,\bar{Q}_k \quad (27b)$$

Being $|E(\Delta t)|$ a diagonal matrix with terms $e^{-\alpha_i \Delta t}$ and $|\Psi||$ a matrix that has to be computed only once

being

$$|\Psi| = \left||I| - |E(\Delta t)|\right||\alpha^{-1}|\left|A'\right| \quad (28)$$

Notice the equivalence between the analitical and the discrete formulacions, eigenfuncions versus eigenvectors, or (9), (10), (13), (14) … versus (23), (24), (26), (27)…. This indicates the identical methodologies used.

3.1 Basic Actions and Control Variables

Alternatives to simulate aquifers commonly do not imply a detailed assignation of stresses on each cell. Instead, alternatives are defined for pumping or recharge in larger areas, although they may include some point actions, [7]. In this case, it is possible to express these actions with a small number of basis vectors by introducing

$$\bar{Q}_k = |Q^e|\overline{IN}_k \quad (29)$$

Matrix $|Q^e|$ with dimensions N by E, contains E unitary basic vectors, and $\overline{IN}_k$ is a column vector with the intensities of the elementary actions. So the following expression is obtained.

$$\bar{L}_k = |E(\Delta t)|\bar{L}_{k-1} + |\Psi|\overline{IN}_k \quad (30)$$

Where $|\Psi||$ is$|\phi|$ $|Q^e|$, being $|Q^e|$ a matrix of dimensions N by E that can also be calculated in advance. Vector $\overline{IN}_k$ is a column vector with the intensities of the elementary actions. More usually E is much less than N so the number of computations to perform is much lower. Additionally, almost never is necessary to know all piezometric heads during all time periods. That represents an excessive computing effort. This is avoided by calculating the heads only in the cells and on times required. Once the vector L is known, the expression (23) is used to calculate the piezometric head in some cells, to calculate the volumes above the zero level in all or part of the aquifer, or the aquifer output to a stretch of river or to a spring. That can be done deleting the unnecessary rows and including new ones using linear combinations of rows multiplied by appropriate transmissivity and storage values. Matrix $|A|$ is transformed into a reduced matrix $|A^R|$of dimension N by M, which may have a number of rows M << N. Thus the control variable vector $\overline{C^k}$ at instant k is obtained with:

$$C^k = |A^R|.\bar{L}_k \quad (31)$$

The method has the great advantage of being explicit and also does not need to store neither influence functions nor actions previously applied to the aquifer. The vector $\bar{L}_k$ is a state vector and equations (27) or (30) are explicit state equations that solve the flow equation sequentially and can be integrated directly in a conjunctive use management model.

In the Plana de Castellón aquifer (eastern Spain), of 450 km^2 and 240 cells, the number of basic stresses used was 25: eleven correspond to distributed recharge or pumping in zones, two to recharge from seepage in reservoirs, two to recharge from streambed infiltration, three to artificial recharge facilities, four to important pumping at specific points and one to lateral recharges and non irrigated zone infiltration. The number of control variables used was 24: eleven correspond to piezometric heads in selected cells of the aquifer, eight correspond to the volume of water over sea level in selected zones of the aquifer, one to the total volume in storage in the aquifer over sea level, and four correspond to the flow to or from the sea level through coastal segments. The dimensions of matrix $|\Psi|$ are 25 by 240 and the dimensions of matrix $|A^R|$ are 240 by 24. Two optimization models were used as screening models to obtain operating rules of the system for three scenarios of alternate groundwater surface water irrigation, with a total of 48 major alternatives, [7].

In the eastern Snake River Plain in Idaho there are over 100,000 water right adjudication claims within the basin [18]; so basin management plans that attempt to delineate the impacts of individual groundwater users on individual surface-water users are impractical. Individual assessments of impacts of groundwater use on senior surface-water supplies could result in tens of thousands of evaluations, even only considering steady state evaluation of capture. They propose the use of some twenty zones defined through cluster analysis, and modified to better conform to existing political and administrative units. It seems clear than the use of the eigenvalue approach offers advantageous possibilities in this and similar cases. All the water pumped on each zone can be grouped as a basic action, and influence functions for pumping in each zone can be obtained. It seems possible simulate all different management alternatives considering decisions on each zone with no more than twice the number of basic actions and a relatively reduced number of control variables: piezometric heads on each zone, flow interchange between the aquifer and specific river reaches and stored water volumes in several aquifer zones. As the model has 1083 cells of 5km x 5km, the size of matrices $|\Psi|$ and $|A^R|$ would be at most on the order of 1083 columns by between 100 to 150 rows

3.2 Computation of Influence Functions

Similarly to obtain the influence function of a unitary pumping in a well *p* for a reach of a river, *r* it is not needed to run the model, but obtain the vector $L_p(t)$ through

$$l_{i,p} = \frac{(1-e^{-\alpha_i t})}{\alpha_i} a_{p,i} \tag{32}$$

Being $a_{p,i}$ the corresponding element of matrix $|A|$.
The influence function in the river reach *r* would be for every time t is

$$q_r^p = \sum_1^N l_{i,r} a_{r,i}, \tag{33}$$

To calculate the influence function of a unitary basic stress, $(d_{1,e} d_{2,e}, \dots\ d_{N,e})$, the vector $L_p(t)$ is

$$l_{i,d} = \frac{1-e^{-\alpha_i t}}{\alpha_i} \sum_j^N a_{j,i}\, d_{j,i} \tag{34}$$

And the influence function would be

$$q_r^d = \sum_1^N l_{i,d} a_{r,i} \tag{35}$$

Even the *influence function of a cyclic action* can be calculated. For example, for an irrigation pumping distributed annually in twelve months according to a coefficient c_i, such that $\sum_1^{12} c_i = 1$, it is easy to determine the influence function for the year *n* and month *m*. If we start the cyclic pumping in month zero of the year 0, at the end of the month m, for a unitary pumping

$$l_{i,p}^{0,m} = \frac{1-e^{-\alpha_i \Delta t}}{\alpha_i} \sum_{k=1}^m c_k e^{-(m-k)\alpha_i \Delta t} \tag{36}$$

Being *i* the eigenvalue mode, *p* the action, *n* the year (for this case zero), and *m* the month.
The dimension of eigenvalue α_i is month^{-1}. For an annual pumping of B

$$l_{i,p}^{0,12} = B\frac{1-e^{-\alpha_i \Delta t}}{\alpha_i} \sum_1^{12} c_k e^{-(12-k)\alpha_i \Delta t} \tag{37}$$

At the end of the year, with a constant value for all the $c_i = 1/12$ monthly values, the l_i for a pumping value of B´ would be $\frac{B'}{12\alpha_i}(1 - e^{-12\alpha_i \Delta t})$, that must be equal to (37); notice that

12 α_i expressed inmonth^{-1}, is equal to α_i expressed in year^{-1}. The value of B´ would be very close, but not equal to B, so B´ should be computed from

$$\frac{B'}{12\alpha_i}\left(1 - e^{-12\alpha_i \Delta t}\right) = l_{i,p}^{0,12} = B\,\frac{1-e^{-\alpha_i \Delta t}}{\alpha_i}\,\sum_1^{12} c_k e^{-(12-k)\alpha_i \Delta t} \quad (38)$$

So the influence function for a cyclic unitary pumping would be deduced from the components of vector $L^{n,m}$

$$l_{i,p}^{n,m} = \frac{B'}{B}\,\frac{\left(1-e^{-\alpha_i 12 n \Delta t}\right)}{12\alpha_i}\,a_{p,i} + \frac{1-e^{-\alpha_i \Delta t}}{\alpha_i}\,\sum_{k=1}^{m} c_k e^{-(m-k)\alpha_i \Delta t} \quad (39)$$

3.2.1 Nonlinearity

Both the application of the influence function method or the eigenvalues method requires that the system behaves linearly or almost linearly. The nonlinearity may occur in the following situations:

a) Relatively large variation in the saturated thickness,
b) Change in the river-aquifer relationship: changing the river from connected and looser to disconnected - the shower effect of hydrologists - or connecting a river reach previously disconnected after recharge increases,
c) Drying of a spring, or his reappearance if levels rise
d) Not having enough flow in the river to provide the required infiltration to an aquifer whose levels are below the riverbed.

Methods have been developed to address these problems. Exploited aquifers connected with the surface system are mostly unconfined, and linear models are not adequate when significant water level changes exist. A two-step explicit method to liberalize the Boussinesq equation based on the eigenvalue approach has been developed. Using a change of variable, it is possible to define an equation with a structure similar to the linear groundwater flow equation. The only difference is found in a term that depends on the initial solution. Approaching this term by means of a fictitious stress a linear equation analogous to the confined groundwater flow equation is obtained [19]. So the problem is solved by applying the superposition principle with a reduced computational cost. By the moment it has been only applied to synthetic examples. Nonlinearities due to changes of the configuration of boundary conditions, as in the cases b) to d) above, have been solved very often using a fictitious action to compensate for deviations between the linear model and the actual situation. Application of this procedure conserves the computational advantages of the Eigenvalue Method. It is based on correcting the non-linear boundary conditions by superposing fictitious stresses in the cells where these boundary conditions are modeled. The intensities of these additional stresses are obtained by solving a system of linear equations. These equations are defined by specifying that the volume exchange in each of these cells is equal to what would exist under non-linear boundary conditions. The methodology has been applied to simulate the groundwater flow of the ‘‘Molar” and ‘‘Vega Alta” aquifers in the Segura River Basin (south-eastern Spain), and the accuracy of the results has been demonstrated through comparison with those obtained using MODFLOW [20].

The last cases, b) to d), need to implement a checking device to test if nonlinear situations are being produced, as connection or disconnection, spring drying or re-appearing, or lack of river flow. In those cases, additional head variations in several cells need to be computed in the eigenvalue simulation process.

4. IMPROVEMENTS AND RESEARCH NEEDS

The eigenvalue method presents significant advantages in the analysis of the conjunctive use, particularly in cases with multiple alternatives and large simulation periods. It has been implemented in AQUATOOL, a generalized Decision Support System (DSS) developed at the Polytechnic University of Valencia (UPV), Valencia- Spain. The DSS was designed for the planning stage of complex basins, including multiple reservoirs, aquifers and demand centers. Up to now the more complex system analyzed in terms of the number of aquifers is the Segura basin, in south-western Spain. It includes some 15 reservoirs, 18 inflows, 93 channels, 50 demand centers, four hydropower plants, 19 aquifers and five additional pumping stations [21]. Until now the code AQUIVAL included in the SDS AQUATOOL has not been used to obtain the eigenvalues and eigenfunctions of aquifers with more than several hundred nodes.

Another aspect that needs improvement in the eigenvalue method to reduce the computation time and the needs of storage, is the elimination of modes with less significance. So far, it has been done excluding the eigenvalues which exceed a certain value [22] so-called traumatic truncation, but the goal is to make a reduction that explicitly considers the influence on the error involved in the elimination of a large number of nodes. AQUIVAL uses QL algorithm with implicit shifts to calculate all eigenvalues and eigenvectors using Jacobi and Givens rotations. If the size of the model to be solved with eigenvalue method is too big, the QL algorithm is inefficient. Álvarez-Villa, in his Ph.D. dissertation, (under review), has implemented efficient algorithms to solve the generalized eigenvalue problems and has developed programs to truncate highly discretized models to gain efficiency without losing accuracy. Numerical experiments have been performed using a highly heterogeneous aquifer. The groundwater flow equation has been solved using finite differences and the eigenvalue method: 22000 blocks have been used to discretize aquifer´s spatial domain and the simulation horizon consisted of 5113 days. The flow simulation lasted about 5.5 hours when finite differences were used. The disperse linear system involved was solved via incomplete LU preconditioned conjugate gradient. When eigenvalues method is used, the simulation lasted about 4 minutes. The calculation of the 500 eigenvalues used was performed in about 2 hours via the rational Lanczos method [23]. Both simulations were performed on an Intel core i7 processor.

5. CONCLUSIONS

The eigenvalue method is a general solution procedure applicable to lineal groundwater models. Piezometric heads can be presented as the L vector of components in the eigenvector´s orthogonal base $|A|$ of an algebraic eigenproblem that is computed once. The result of any external stress on the aquifer is obtained directly and explicitly by a simple equation of state, which involves the eigenvectors of the matrix A, the eigenvalues α_i of the eigenproblem and the vector of external stresses. The operation that requires more computational effort is the determination of the eigenvalues and eigenvectors. Instead, it is very easy to calculate the response of the aquifer to external actions and is made directly and explicitly through a very simple state equation. So the method does not need to store influence functions or actions previously applied to aquifer. It is more useful the greater is the overall number of time steps in all simulations. If the total number of increments of time to perform all simulations is small, obtaining the eigenvalues and eigenvectors may not be appropriate and would result costly for problems with high number of cells.

An important advantage is that almost never is necessary to know all piezometric heads during all time periods. That represents an excessive computing effort. This is avoided by calculating the heads only in the cells and on times required. Once the vector L is known, the expression (31) is used to calculate the piezometric head in some cells, to calculate the volumes above the zero level in all or part of the aquifer, or the aquifer output to a reach of river or to a spring. That can be done deleting the unnecessary rows and including new ones using linear combinations of rows multiplied by appropriate transmissivity and storage values. Matrix $|A|$ is transformed into a reduced matrix $|A^R|$ to obtain a control variable $\overline{C^k}$ with a number of rows much smaller than N.

Until now, the eigenvalue technology has been applied to aquifers with some hundreds of cells, but there is a need of efficient algorithms to solve the generalized eigenvalue problems of several thousands of nodes. Another aspect that needs improvement in the eigenvalue method to reduce the computational burden and the needs of storage is the elimination of unnecessary modes. Up to now, it has been done aplying the traumatic truncation. The progress made in the above-mentioned ph dissertation of Alvarez Villa to efficiently determine only dominant modes appears to be an interesting step for the management of aquifers with high number of nodes. Similarly when is needed to obtain hundreds or thousands of influence functions for many values of time the application of the formulas presented in 3.2 may be much more convenient that performing thousands of runs of the model. And similarly it may also be interesting its application in optimization models for linear programming, dynamic programming or other optimization methods.

ACKNOWLEDGEMENTS

The paper is a product of the research on the topic of conjunctive use modeling developed by the Department of Water and Environmental Engineering of the Polytechnic University of Valencia, Spain during the last thirty years, being many the actual contributors to the reseach. I would like to thank Eduardo Cassiraga and Amanda Mejia for their assistance in preparing figures and tables, to Oscar Alvarez-Villa for the comments on his ph. dissertation, to Jaime Gomez-Hernandez and Jim Butler for their observations and comments, and especially to Emilio Custodio and Joaquin Andreu for their detailed review and constructive comments.

SYMBOLS

T_x, T_x (x, y) and T_y, T_y(x, y)	L^2/T	principal components of transmissivity tensor
S	L^0	storage coefficient,
Q (x, y), Q_d(x, y), Q_i(x, y)	L/T, L^3/T	stress
h, w, s	L	piezometric head, hydraulic potential
$P_k(m\Delta t)$	L^0	magnitude of stress k
$\delta_k^i(n\Delta t)$	L,	influence of a k unit stress
$Q_{inf}^j(n\Delta t)$, $Q_{nat}^j(n\Delta t)$	L^3/T, L^3/T	influenced, or natural flow,
A_i (x, y), $A_{m,n}$ (x ,y)	L^{-1}	eigenfunction
α_i	T^{-1}	eigenvalue

l_i	L^2	components of vector L of eigenfunctions or eigenvectors
b_i	L^0	partition coefficient
F_i	L	volume under the surface of the eigenfunction A_i.
λ	L^0	dimensionless hydraulic connection parameter
$\lvert T \rvert$	L^2/T	banded transmissivity matrix
$\lvert SF \rvert$	L^2	cell storage diagonal matrix
$\lvert A \rvert$	L^2	eigenvectors matrix
$\lvert \alpha \rvert$	T^{-1}	eigenvalues diagonal matrix
$\lvert A^R \rvert$	L^2	reduced eigenvectors matrix

COMPETING INTERESTS

Author has declared that no competing interests exist.

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