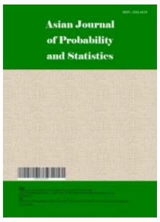

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Max Weibull-G Power Series Distributions

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

Based on the Weibull-G Power probability distribution family, we have proposed a new family of probability distributions, named by us the Max Weibull-G power series distributions, which may be applied in order to solve some reliability problems. This implies the fact that the Max Weibull-G power series is the distribution of a random variable $\max(X_1, X_2, \ldots X_N)$ where $X_1, X_2, \ldots$ are Weibull-G distributed independent random variables and N is a natural random variable the distribution of which belongs to the family of power series distribution. The main characteristics and properties of this distribution are analyzed.

Keywords: Lifetime; power series distribution; distribution of the maximum; Weibull distribution.

1 Introduction

The Weibull distribution is often found in reliability modeling and not only. It is mainly used in the rate of stochastical modeling hazard when analyzing the failure rate in electrical engineering, or studying the super tension occurring in a circuit in industrial engineering within the study of delivery times, the extreme values, the weather forecast, the radar systems for modeling the signals level dispersion received from different sources, etc.

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Additionally, large-scale applications in various fields, such as engineering, lifetime, and economy, etc. have led to the introduction of a new family of distribution models, the Weibull-G (NWG) [1]. The well-known generators are, as follows: beta-G by Eugene et al. [2], Kumaraswamy-G (Kw-G) by Cordeiro and de Castro [3], McDonald-G (Mc-G) by Alexander et al. [4], gamma-G type1 by Zografos and Balakrishanan [5] and Amini et al. [6], gamma-G type 2 by Ristić and Balakrishanan [7] and Amini et al. [6], odd exponentiated generalized (odd exp-G) by Cordeiro et al. [8], transformed-transformer (T-X) (Weibull-Xand gamma-X) by Alzaatreh et al. [9], exponentiated T-X by Alzaghal et al. [10], odd Weibull-G by Bourguignon et al. [11], exponentiated half-logistic by Cordeiro et al. [12], the T-X{Y}-quantile based approach by Aljarrah et al. [13], T-R{Y} by Alzaatreh et al. [14], Lomax-G by Cordeiro et al. [15], logistic-X by Tahir et al. [16] and Kumaraswamy odd log-logistic-G by Alizadeh et al. [17].

The distribution class of the Max Weibull-G power series comes to complete the research with reference to the distribution family generated by the Weibull distribution. In this case, the treatment is done unitarily using the power series distribution class (PSD).

In this paper the distribution for the maximum of Z independent and identically Weibull-G distributed random variables are analysed, where number Z is a power series distributed random variable.

The article is structured as follows: in section 2, preliminary results are presented, including the basic notions of power series distributions, examples of distributions of this type where the power series, the power series parameter and the corresponding convergence radius are highlighted and, Finally, the distribution function and the probability density of the Weibull G distribution are defined. The main features and properties of the Max Weibull G distribution are analyzed in section 3. In section 4, the Poisson limit theorem is formulated and proved in the context of the Max Weibull G distribution. The validation of the Poisson limit theorem from a graphical point of view is completed in section 5 for some special cases of Max Weibull G distributions. Section 6 of the paper represents the conclusions to the study.

2 Preliminary Results

Over the years, many researchers have created reliability models which generate certain probability distribution families, as they create a good flexibility when modeling and analyzing data in practice. For example, Gurvich [18] introduces a model which is based on the Weibull distribution: $G(x,\alpha,\boldsymbol{\eta}) = 1-\exp(-\alpha H(x,\boldsymbol{\eta}))$ **,** $x \in D \subseteq R$ **,** $\alpha > 0$, where $H(x,\boldsymbol{\eta})$ is a nonnegative, increasing function, which depends on the vector parameter and this distribution generalizes the exponential distribution $H(x,\boldsymbol{\eta}) = x$, Rayleigh ($H(x,\boldsymbol{\eta}) = x^2$), Pareto ($H(x,\boldsymbol{\eta}) = \ln\frac{x}{k}$) .

Then Zografos and Balakrishman [5] define a new distribution family starting from the probability density of Stacy's generalized gamma distribution.

Recently, Bourguignon et al. [6] has generated the Weibull distribution families introduced by Gurvich [18] and Zografos and Balakrishman [6], thus presenting a new Weibull distributions family using the Weibull generator applied to the ratio $G(x)/\overline{G}(x)$, where $\overline{G}(x) = 1-G(x)$ **.** Thus, in the cumulative distribution function of the Weibull distribution with real parameters and positive .

one ($F(x) = 1-\exp\left[-\left(\frac{x}{\lambda}\right)^{\nu}\right]$, $x > 0$**,** $\lambda,\upsilon > 0$), replacing x by , we obtain the distribution function family for the Weibull-G distribution class:

$$F_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta}) = 1-\exp\left\{-\frac{1}{\lambda^{\upsilon}}\left[\frac{G(x,\boldsymbol{\eta})}{\overline{G}(x,\boldsymbol{\eta})}\right]^{\upsilon}\right\}, \tag{1}$$

$x \in D \subseteq R^*$, $\lambda, \upsilon > 0$ and $G(x, \boldsymbol{\eta})$ is the distribution function referred to, which depends on the vector parameter. Also, the probability density function which characterizes this distribution family is:

$$f_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta}) = \frac{\upsilon}{\lambda^{\upsilon}} g(x, \boldsymbol{\eta}) \frac{G^{\upsilon-1}(x, \boldsymbol{\eta})}{\overline{G}^{\upsilon+1}(x, \boldsymbol{\eta})} \exp\left\{ -\frac{1}{\lambda^{\upsilon}} \left[\frac{G(x, \boldsymbol{\eta})}{\overline{G}(x, \boldsymbol{\eta})} \right]^{\upsilon} \right\}, \quad (2)$$

where $g(x, \boldsymbol{\eta})$ is the probability density function corresponding to the absolutely continuous type distribution G.

Indeed,

$$f_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta}) = \frac{d}{dx} F_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta}) = \frac{\upsilon}{\lambda^{\upsilon}} \left(\frac{G(x,\boldsymbol{\eta})}{\overline{G}(x,\boldsymbol{\eta})} \right)^{\upsilon-1} \exp\left\{ -\frac{1}{\lambda^{\upsilon}} \left[\frac{G(x,\boldsymbol{\eta})}{\overline{G}(x,\boldsymbol{\eta})} \right]^{\upsilon} \right\} \cdot \frac{d}{dx} \left(\frac{G(x,\boldsymbol{\eta})}{1-G(x,\boldsymbol{\eta})} \right)$$

$$= \frac{\upsilon}{\lambda^{\upsilon}} \left(\frac{G(x,\boldsymbol{\eta})}{\overline{G}(x,\boldsymbol{\eta})} \right)^{\upsilon-1} \exp\left\{ -\frac{1}{\lambda^{\upsilon}} \left[\frac{G(x,\boldsymbol{\eta})}{\overline{G}(x,\boldsymbol{\eta})} \right]^{\upsilon} \right\} \cdot \frac{\frac{d}{dx} G(x,\boldsymbol{\eta}) \cdot G^{\upsilon-1}(x,\boldsymbol{\eta})}{\overline{G}^{2}(x,\boldsymbol{\eta})}$$

$$= \frac{\upsilon}{\lambda^{\upsilon}} \exp\left\{ -\frac{1}{\lambda^{\upsilon}} \left[\frac{G(x,\boldsymbol{\eta})}{\overline{G}(x,\boldsymbol{\eta})} \right]^{\upsilon} \right\} \cdot \frac{g(x,\boldsymbol{\eta}) \cdot G^{\upsilon-1}(x,\boldsymbol{\eta})}{\overline{G}^{\upsilon+1}(x,\boldsymbol{\eta})}, \text{ unde } g(x,\boldsymbol{\eta}) = \frac{d}{dx} G(x,\boldsymbol{\eta}).$$

A random variable which has the cumulative distribution function defined by Eq. (1) or the probability density function defined by Eq. (2) is denoted $X \sim WG(\lambda, \upsilon, \boldsymbol{\eta})$.

Bourguignon et al. [6] give the following interpretation to the Weibull-G distribution family, that is: let Y be the lifetime with the absolutely continuous type G distribution. The chance ratio for a component with a lifetime Y to degrade to the x moment is namely it represents exactly the distribution of the random variable X:

$$\mathbf{P}\left(X \le \frac{G(x, \boldsymbol{\eta})}{\overline{G}(x, \boldsymbol{\eta})} \right) = \mathbf{P}(Y \le x) = F_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta}),$$

where $F_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta})$, $x > 0$, $\lambda, \upsilon > 0$ is defined by Eq. (1) and $G(x, \boldsymbol{\eta})$ is the absolutely continuous distribution function type which depends on the vector parameter $\boldsymbol{\eta}$.

Consequently, let us reproduce the definition of the power series distributions used by us, considering random variable Z such that $\mathbf{P}(Z \in \{1,2,\ldots\}) = 1$.

Definition 2.1. ([19]) We say that the random variable Z has a power series distribution if:

$$\mathbf{P}(Z = z) = \frac{a_z \Theta^z}{A(\Theta)}, \ z = 1,2,\ldots; \ \Theta \in (0, \tau); \ \tau > 0,$$

(3)

where $a_1, a_2, \ldots$ are nonnegative real numbers, τ is a positive number bound by the convergence radius of power series (series function) $A(\Theta) = \sum_{z \ge 1} a_z \Theta^z, \ \forall \ \Theta \in (0, \tau)$ and is the power parameter of the distribution.

Remark 2.1. Let us denote by PSD the class of the power series distributions. Some of them may be found in Table 1. The fact that the random variable Z has the distribution Eq. (3) will be denoted as $Z \in PSD$.

Table 1. The representative elements of the PSD class for various truncated distributions (Binomial, Poisson, Log, Geometric, Pascal, Bernoulli distributions)

Distributions	a_z	**Power parameter** θ	**Series function** $A(\theta)$	**Convergence radius** τ
$Binom^*(n,p)$	$\binom{n}{z}$	$\frac{p}{1-p}$	$(1+\theta)^n - 1$	∞
$Poisson^*(\lambda)$	$\frac{1}{z!}$	λ	$e^\theta - 1$	∞
$Log(p)$	$\frac{1}{z}$	p	$-\ln(1-\theta)$	1
$Geom^*(p)$	1	$1-p$	$\frac{\theta}{1-\theta}$	1
$Pascal(k,p)$	$\binom{z-1}{k-1}$	$1-p$	$\left(\frac{\theta}{1-\theta}\right)^k$	1
$BN^*(k,p)$	$\binom{z+k-1}{z}$	p	$(1-\theta)^{-k} - 1$	1

3 Max Weibull-G Power Series Distribution

Let be a sequence of independent and identically distributed random variables, $X_i \sim WG(\lambda, \upsilon, \boldsymbol{\eta})$, $\lambda, \upsilon > 0$ and a vectorial parameter of the absolutely continuous type distribution G. We denote by $F_{X_i}(x, \lambda, \upsilon, \boldsymbol{\eta}) \equiv F_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta})$, $x > 0$ the cumulative distribution function respectively $f_{X_i}(x, \lambda, \upsilon, \boldsymbol{\eta}) \equiv f_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta})$, $x > 0$ the probability density function of the random variable $(X_i)_{i \geq 1}$, defined by the Eq. (1) and Eq. (2).

We also denote by $U_{WG} = \max\{X_1, X_2, \ldots, X_Z\}$, where the random variable $Z \in PSD$.

According to Leahu et al. [20] the cumulative distribution function cdf and the probability density function of the random variable U_{WG} are characterized by the following relations:

$$U_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta}, \Theta) = \frac{A[\Theta F_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta})]}{A[\Theta]}, \, x > 0 \tag{4}$$

and

$$u_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta}, \Theta) = \frac{\Theta f_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta}) A'[\Theta F_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta})]}{A[\Theta]}, \, x > 0 \tag{5}$$

Definition 3.1. We say that U_{WG} has a Max Weibull-G power series distribution with parameters $\lambda, \upsilon, \boldsymbol{\eta}$ (denoted $U_{WG} \sim MaxWGPS(\lambda, \upsilon, \boldsymbol{\eta}, \Theta)$) if this has the cumulative distribution function defined by the Eq. (4) or the probability density function defined by Eq. (5), where $A(\Theta)$ is the power series which depends on the power parameter which relates to the distribution of the random variable Z.

The following results characterize the function of survival and the rate of hazard.

Consequence 3.1. The function of survival of the random variable U_{WG} is the following:

$$S_{U_{WG}}(x,\lambda,\upsilon,\boldsymbol{\eta},\Theta)=1-\frac{A[\Theta F_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta})]}{A[\Theta]},\ x>0.$$

Consequence 3.2. The rate of hazard for the random variable U_{WG} is given by:

$$h_{U_{WG}}(x,\lambda,\upsilon,\boldsymbol{\eta},\Theta)=\frac{u_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta},\Theta)}{S_{U_{WG}}(x,\lambda,\upsilon,\boldsymbol{\eta},\Theta)}=\frac{\Theta f_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta})A'[\Theta F_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta})]}{A[\Theta]-A[\Theta F_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta})]},\ x>0.$$

The following result presents some properties of distribution for the maximum of random number of independent and identically distributed random variables with the cumulative distribution function $F_{X_i}(x,\lambda,\upsilon,\boldsymbol{\eta})\equiv F_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta})$, $x>0$.

Proposition 3.1. If $(X_i)_{i\geq 1}$ is a sequence of nonnegative independent and identically distributed random variables of absolutely continuous type with the cumulative distribution function $F_{X_i}(x,\lambda,\upsilon,\boldsymbol{\eta})\equiv F_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta})$, $x>0$ and with $Z\in PSD$ $P(Z=z)=\frac{a_z\Theta^z}{A(\Theta)}$, $(a_z)_{z\geq 1}$ a sequence of real, nonnegative numbers, $A(\Theta)=\sum_{z\geq 1}a_z\Theta^z$, $\forall\ \Theta\in(0,\tau)$, then:

$$\lim_{\Theta\to 0^+}U_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta},\Theta)=[F_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta},\Theta)]^k,\ x>0,$$

where $k=\min\{n\in N^*, a_n>0\}$.

The expression for the moment of r^{th} order of the random variable U_{WG} is presented further on.

Proposition 3.2. The moment of r^{th} order $r\in N$, $r\geq 1$ of the random variable $U_{WG}=\max\{X_1,X_2,\ldots,X_Z\}$ where $Z\in PSD$, is characterized by the relation:

$$EU_{WG}^r=\sum_{z\geq 1}\frac{a_z\Theta^z}{A(\Theta)}E[\max\{X_1,X_2,\ldots,X_z\}]^r, \tag{6}$$

where the probability density function of the random variable $\max\{X_1,X_2,\ldots,X_z\}$ is:

$$f_{\max\{X_1,X_2,\ldots X_z\}}(x,\lambda,\upsilon,\boldsymbol{\eta})=zf_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta})[F_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta})]^{z-1}$$

Other properties of the Max Weibull-G power series distributions are given by the following proposition that shows that the distribution of the maximum of the first k random variable of a countable sequence of independent and identically distributed random variables is, in certain conditions, equivalent to the distribution of a sequence of random variables in a random number.

The following notations are required: $U_{WGGeom}=\max\{X_1,X_2,\ldots,X_Z\}$, where $Z\sim Geom(p)$, $p\in(0,1)$ and $X_i, i\geq 1$ are independent and identically distributed random variables with the cumulative distribution function F_{WG}.

Definition 3.2. We say that U_{WGGeom} has a Max Weibull-G geometrical power series distribution of $\lambda, \upsilon, \boldsymbol{\eta}$ and p parameters, where $\lambda, \upsilon > 0$ refers to Weibull distribution parameters, η the vector parameter of the absolutely continuous distribution G and $p \in (0,1)$ geometrical distribution parameter (noted by $U_{WGGeom} \sim MaxWGGeom(\lambda, \upsilon, \boldsymbol{\eta}, p)$), if it has the cumulative distribution function defined by Eq.(4) or the probability density function defined by Eq. (5), where $\Theta = 1 - p,\ p \in (0,1)$ and $A(\Theta) = \dfrac{\Theta}{1-\Theta}$.

Proposition 3.3. If $(X_i)_{i\geq 1}$ are nonnegative independent and identically distributed random variables of the absolutely continuous type and $(Y_j)_{j\geq 1}$ are independent and identically distributed random variables, $Y_j \sim MaxWGGeom(\lambda, \upsilon, \boldsymbol{\eta}, p)$, $\lambda, \upsilon > 0$, $p \in (0,1)$ and $\boldsymbol{\eta}$ vector parameter of the G absolutely continuous distribution, then the random variable has the same distribution as the random variable $\max\{X_1, X_2, \ldots, X_Z\}$, where $Z \sim Pascal(k, p)$, $k \in \{1, 2, \ldots\}$, $p \in (0,1)$.

4 Poisson Limit Theorem

Let's denote $U_{WGBinom} = \max\{X_1, X_2, \ldots, X_Z\}$ and $U_{WGPoisson} = \max\{X_1, X_2, \ldots, X_N\}$, where $(X_i)_{i\geq 1}$ are random variable nonnegative, independent and identically distributed with the the cumulative distribution function $F_{X_i}(x, \lambda, \upsilon, \boldsymbol{\eta}) \equiv F_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta})$, $\forall x > 0$, $Z \sim Binom^*(n, p)$, $n \in \{1, 2, \ldots\}$, $p \in (0,1)$ with $A(\Theta) = (1+\Theta)^n - 1$, $\Theta \in (0, \infty)$, $\Theta = \dfrac{p}{1-p}$ and $N \sim Poisson^*(\alpha)$, $\alpha > 0$, with $A(\Theta^*) = \exp(\Theta^*) - 1$, $\Theta^* \in (0, \infty)$, $\Theta^* = \alpha$, the random variable $(X_i)_{i\geq 1}$ and Z, respectively $(X_i)_{i\geq 1}$ and N are reciprocally independent.

Now we can formulate the following definitions.

Definition 4.1. (i) We say that random variable $U_{WGBinom}$ has a Max Weibull-G binomial power series distribution of parameters $\lambda, \upsilon, \boldsymbol{\eta}$ and p (noted $U_{WGBinom} \sim MaxWGBinom(\lambda, \upsilon, \eta, n, p)$) if this has the cumulative distribution function defined by the relation:

$$U_{WGBinom}(x, \lambda, \upsilon, \boldsymbol{\eta}, n, p) \overset{(4)}{=} \frac{\left\{1 - p \cdot \exp\left[-\left(\dfrac{G(x, \boldsymbol{\eta})}{\lambda \overline{G}(x, \boldsymbol{\eta})}\right)^{\upsilon}\right]\right\}^{n} - (1-p)^n}{1 - (1-p)^n}, \tag{7}$$

$x > 0$, $\lambda, \upsilon > 0$, $n \in \{1, 2, \ldots\}$, $p \in (0,1)$, or the probability density function characterized by:

$$u_{WGBinom}(x, \lambda, \upsilon, \boldsymbol{\eta}, n, p) \overset{(5)}{=}$$

$$= \frac{np\upsilon g(x, \boldsymbol{\eta}) \left(\dfrac{G(x, \boldsymbol{\eta})}{\lambda \overline{G}(x, \boldsymbol{\eta})}\right)^{\upsilon} \exp\left[-\left(\dfrac{G(x, \boldsymbol{\eta})}{\lambda \overline{G}(x, \boldsymbol{\eta})}\right)^{\upsilon}\right] \left\{1 - p \cdot \exp\left[-\left(\dfrac{G(x, \boldsymbol{\eta})}{\lambda \overline{G}(x, \boldsymbol{\eta})}\right)^{\upsilon}\right]\right\}^{n-1}}{G(x, \boldsymbol{\eta}) \overline{G}(x, \boldsymbol{\eta}) \left[1 - (1-p)^n\right]}. \tag{8}$$

(ii) We say that the random variable $U_{WGPoisson}$ has a Max Weibull-G Poisson power series distribution of parameters $\lambda, \upsilon, \boldsymbol{\eta}$ and α (it is noted $U_{WGPoisson} \sim MaxWGPoisson(\lambda, \upsilon, \boldsymbol{\eta}, \alpha)$ if this has the cumulative distribution function defined by the relation:

$$U_{WGPoisson}(x, \lambda, \upsilon, \boldsymbol{\eta}, \alpha) \overset{(4)}{=} \frac{\exp\left\{-\alpha \exp\left[-\left(\dfrac{G(x, \boldsymbol{\eta})}{\lambda \overline{G}(x, \boldsymbol{\eta})}\right)^{\upsilon}\right]\right\} - \exp(-\alpha)}{1 - \exp(-\alpha)}, \tag{9}$$

$x > 0$, $\lambda, \upsilon > 0$, $\alpha > 0$, $p \in (0,1)$ or the probability density function given by the relation:

$$u_{WGPoisson}(x, \lambda, \upsilon, \boldsymbol{\eta}, \alpha) \overset{(5)}{=}$$

$$= \frac{\alpha \upsilon g(x, \boldsymbol{\eta}) \left(\dfrac{G(x, \boldsymbol{\eta})}{\lambda \overline{G}(x, \boldsymbol{\eta})}\right)^{\upsilon} \exp\left\{-\left(\dfrac{G(x, \boldsymbol{\eta})}{\lambda \overline{G}(x, \boldsymbol{\eta})}\right)^{\upsilon} - \alpha \exp\left[-\left(\dfrac{G(x, \boldsymbol{\eta})}{\lambda \overline{G}(x, \boldsymbol{\eta})}\right)^{\upsilon}\right]\right\}}{G(x, \boldsymbol{\eta}) \overline{G}(x, \boldsymbol{\eta}) [1 - \exp(-\alpha)]}. \tag{10}$$

Remark 4.1. (i)The cumulative distribution functions defined by the Eq.(7) and Eq.(9) result from:
and

(ii) The probability density functions defined by the Eq.(8) and Eq.(10) result from:

$$U_{WGBinom}(x, \lambda, \upsilon, \boldsymbol{\eta}, n, p) \overset{(4)}{=} \frac{A\left[\Theta F_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta})\right]}{A[\Theta]} = \frac{\left(1 + \Theta F_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta})\right)^n - 1}{(1 + \Theta)^n - 1}$$

$$= \frac{\left(1 + \dfrac{p}{1-p} F_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta})\right)^n - 1}{\left(1 + \dfrac{p}{1-p}\right)^n - 1} \overset{(1)}{=} \frac{\left\{1 - p \cdot \exp\left[-\left(\dfrac{G(x, \boldsymbol{\eta})}{\lambda \overline{G}(x, \boldsymbol{\eta})}\right)^{\upsilon}\right]\right\}^n - (1-p)^n}{1 - (1-p)^n}$$

and

$$U_{WGPoisson}(x, \lambda, \upsilon, \boldsymbol{\eta}, \alpha) \overset{(4)}{=} \frac{A\left[\Theta^* F_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta})\right]}{A\left[\Theta^*\right]} = \frac{\exp\left[\Theta^* F_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta})\right] - 1}{\exp\left(\Theta^*\right) - 1}$$

$$= \frac{\exp\left[\alpha F_{WG}(x, \lambda, \upsilon, \boldsymbol{\eta})\right] - 1}{\exp\left(\Theta^*\right) - 1} \overset{(1)}{=} \frac{\exp\left\{\alpha - \alpha \cdot \exp\left[-\left(\dfrac{G(x, \boldsymbol{\eta})}{\lambda \overline{G}(x, \boldsymbol{\eta})}\right)^{\upsilon}\right]\right\} - 1}{\exp(\alpha) - 1}$$

$$= \frac{\exp\left\{-\alpha \exp\left[-\left(\dfrac{G(x, \eta)}{\lambda \overline{G}(x, \eta)}\right)^{\upsilon}\right]\right\} - \exp(-\alpha)}{1 - \exp(-\alpha)}.$$

(ii) *The probability density functions* defined by the Eq.(8) and Eq.(10) result from:

$$u_{WGBinom}(x,\lambda,\upsilon,\eta,n,p) \overset{(5)}{=} \frac{\Theta f_{WG}(x,\lambda,\upsilon,\eta)\dfrac{d}{dx}A\left[\Theta F_{WG}(x,\lambda,\upsilon,\eta)\right]}{A\left[\Theta\right]}$$

$$= \frac{\Theta f_{WG}(x,\lambda,\upsilon,\eta)\cdot n\left(1+\Theta F_{WG}(x,\lambda,\upsilon,\eta)\right)^{n-1}}{\left(1+\Theta\right)^{n}-1} = \frac{\dfrac{p}{1-p}\, f_{WG}(x,\lambda,\upsilon,\eta)\cdot n\left(1+\dfrac{p}{1-p}\cdot F_{WG}(x,\lambda,\upsilon,\eta)\right)^{n-1}}{\left(1+\dfrac{p}{1-p}\right)^{n}-1}$$

$$= \frac{np\cdot f_{WG}(x,\lambda,\upsilon,\eta)\cdot\left(1-p+p\cdot F_{WG}(x,\lambda,\upsilon,\eta)\right)^{n-1}}{1-\left(1-p\right)^{n}}$$

$$\overset{(1),(2)}{=} \frac{np\upsilon g(x,\eta)\left(\dfrac{G(x,\eta)}{\lambda\overline{G}(x,\eta)}\right)^{\upsilon}\exp\left[-\left(\dfrac{G(x,\eta)}{\lambda\overline{G}(x,\eta)}\right)^{\upsilon}\right]\left\{1-p\cdot\exp\left[-\left(\dfrac{G(x,\eta)}{\lambda\overline{G}(x,\eta)}\right)^{\upsilon}\right]\right\}^{n-1}}{G(x,\eta)\cdot\overline{G}(x,\eta)\left[1-(1-p)^{n}\right]}$$

and

$$u_{WGPoisson}(x,\lambda,\upsilon,\eta,\alpha) \overset{(5)}{=} \frac{\Theta^{*} f_{WG}(x,\lambda,\upsilon,\eta)\dfrac{d}{dx}A\left[\Theta^{*}F_{WG}(x,\lambda,\upsilon,\eta)\right]}{A\left[\Theta^{*}\right]}$$

$$= \frac{\Theta^{*} f_{WG}(x,\lambda,\upsilon,\eta)\exp\left[\Theta^{*}F_{WG}(x,\lambda,\upsilon,\eta)\right]}{\exp\left(\Theta^{*}\right)-1} = \frac{\alpha f_{WG}(x,\lambda,\upsilon,\eta)\exp\left[\alpha F_{WG}(x,\lambda,\upsilon,\eta)\right]}{\exp\left(\alpha\right)-1}$$

$$\overset{(1),(2)}{=} \frac{\alpha\upsilon g(x,\eta)\left(\dfrac{G(x,\eta)}{\lambda\overline{G}(x,\eta)}\right)^{\upsilon}\exp\left\{-\left(\dfrac{G(x,\eta)}{\lambda\overline{G}(x,\eta)}\right)^{\upsilon}-\alpha\exp\left[-\left(\dfrac{G(x,\eta)}{\lambda\overline{G}(x,\eta)}\right)^{\upsilon}\right]\right\}}{G(x,\eta)\cdot\overline{G}(x,\eta)\left[1-\exp(-\alpha)\right]}.$$

The following result shows that the Max Weibull-G Poisson (with the cumulative distribution function $U_{WGPoisson}(x,\lambda,\upsilon,\eta,\alpha)$) approximates in certain conditions the distribution Max Weibull-G Binomial (with the cumulative distribution function $U_{WGBinom}(x,\lambda,\upsilon,\eta,n,p)$).

Theorem 4.1. (*Poisson limit theorem)* *If the random variable.* $U_{WGBinom} \sim MaxWGBinom(\lambda,\upsilon,\boldsymbol{\eta},n,p)$ *and* $U_{WGPoisson} \sim MaxWGPoisson\,(\lambda,\upsilon,\eta,\alpha)$ *with* $n\to\infty$ *and* $p\to 0^{+}$ *so that* $np\to\alpha,\ \alpha>0$*, then*

$$\lim_{\substack{n\to\infty\\ p\to 0^{+}}} U_{WGBinom}(x,\lambda,\upsilon,\boldsymbol{\eta},n,p) = U_{WGPoisson}(x,\lambda,\upsilon,\boldsymbol{\eta},\alpha),\ \forall x>0,$$

where $U_{WGBinom}(x,\lambda,\upsilon,\boldsymbol{\eta},n,p)$ *and* $U_{WGPoisson}(x,\lambda,\upsilon,\boldsymbol{\eta},\alpha)$ *are the cumulative distribution functions of the random variable* ~~r.v.~~ $U_{WGBinom}$*, respectively* $U_{WGPoisson}$.

Proof.

We examine the convergence in the terms of the maximum distribution $U_{WGBinom}$ *and* $U_{WGPoisson}$ $x > 0$.

It is evident that:

$$\lim_{\substack{n\to\infty \\ p\to 0^+}} (1-p)^n = \lim_{\substack{n\to\infty \\ p\to 0^+}} \left[(1-p)^{-1/p}\right]^{-pn} = e^{-\alpha},$$

$$\lim_{\substack{n\to\infty \\ p\to 0^+}} \left[1-p+p\cdot F_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta})\right]^n$$

$$= \lim_{\substack{n\to\infty \\ p\to 0^+}} \left[\left[1-p+p\cdot F_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta})\right]^{-1/\left[-p+p\cdot F_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta})\right]}\right]^{\left[-p+p\cdot F_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta})\right]\cdot n}$$

$$= e^{-\alpha\cdot\left(1-F_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta})\right)}.$$

Under these conditions, we obtain:

$$\lim_{\substack{n\to\infty \\ p\to 0^+}} U_{WGBinom}(x,\lambda,\upsilon,\boldsymbol{\eta},n,p) = \lim_{\substack{n\to\infty \\ p\to 0^+}} \frac{\left[1-p+p\cdot F_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta})\right]^n - (1-p)^n}{1-(1-p)^n}$$

$$= \frac{e^{-\alpha\cdot\left(1-F_{WG}(x,\lambda,\upsilon,\boldsymbol{\eta})\right)} - e^{-\alpha}}{1-e^{-\alpha}} = U_{WGPoisson}(x,\lambda,\upsilon,\boldsymbol{\eta},\alpha).$$

5. Special Cases

5.1 Max weibull uniform binomial distribution

In the first example, we consider that distribution G is the uniform distribution U on the interval $(0,\infty)$. Then *the probability density function* is $g(x,\phi) = \frac{1}{\phi}$, $x \in (0,\phi)$ and *the cumulative distribution functions* $G(x,\phi) = \frac{x}{\phi}$.

Max Weibull-Uniform binomial distribution has the *cumulative distribution functions* given by the relation:

$$U_{WUBinom}(x,\lambda,\upsilon,\phi,n,p) \overset{(7)}{=} \frac{\left\{1-p\exp\left[-\left(\frac{x}{\lambda(\phi-x)}\right)^{\upsilon}\right]\right\}^n - (1-p)^n}{1-(1-p)^n}, \; x \in (0,\phi),$$

where $\lambda,\upsilon > 0$, $n \in \{1,2,\ldots\}$, $p \in (0,1)$. The corresponding *probability density function* is:

$$u_{WUBinom}(x,\lambda,\upsilon,\phi,n,p) \overset{(8)}{=} \frac{np\upsilon\phi x^{\upsilon-1}\exp\left[-\left(\frac{x}{\lambda(\phi-x)}\right)^{\upsilon}\right]\left\{1-p\exp\left[-\left(\frac{x}{\lambda(\phi-x)}\right)^{\upsilon}\right]\right\}^{n-1}}{\lambda^{\upsilon}(\phi-x)^{\upsilon+1}\left[1-(1-p)^{n}\right]},$$

with $x \in (0,\phi)$.

5.2 Max Weibull uniform poisson distribution

In the second example, distribution G is the same as uniform U distribution on the interval $x \in (0,\phi)$. Therefore Max Weibull-Uniform Poisson has the *cumulative distribution functions* given by the relation:

$$U_{WUPoisson}(x,\lambda,\upsilon,\phi,\alpha) \overset{(9)}{=} \frac{\exp\left\{-\alpha\exp\left[-\left(\frac{x}{\lambda(\phi-x)}\right)^{\upsilon}\right]\right\}-\exp(-\alpha)}{1-\exp(-\alpha)},\ x \in (0,\phi),$$

where $\lambda,\upsilon > 0$, $\alpha > 0$ and the corresponding *probability density function* is:

$$u_{WUPoisson}(x,\lambda,\upsilon,\phi,\alpha) \overset{(10)}{=} \frac{np\upsilon\phi x^{\upsilon-1}\exp\left\{-\left(\frac{x}{\lambda(\phi-x)}\right)^{\upsilon}-\alpha\exp\left[-\left(\frac{x}{\lambda(\phi-x)}\right)^{\upsilon}\right]\right\}}{\lambda^{\upsilon}(\phi-x)^{\upsilon+1}\left[1-\exp(-\alpha)\right]},$$

with $x \in (0,\phi)$.

5.3 Max Weibull - Weibull binomial distribution

In this example we consider that distribution G is the same as Weibull distribution with the *probability density function* and the *cumulative distribution functions* characterized by the relations: $g(x,a,b) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1}\cdot\exp\left[-\left(\frac{x}{a}\right)^{b}\right]$, respectively $G(x,a,b) = 1-\exp\left[-\left(\frac{x}{a}\right)^{b}\right]$, $a,b > 0$. In these conditions the *cumulative distribution functions* and the *probability density function* which characterize the Max Weibull - Weibull Binomial distribution are the following:

$$U_{WWBinom}(x,\lambda,\upsilon,a,b,n,p) \overset{(7)}{=} \frac{\left\{1-p\exp\left[-\left(\frac{\exp\left(x/a\right)^{b}-1}{\lambda}\right)^{\upsilon}\right]\right\}^{n}-(1-p)^{n}}{1-(1-p)^{n}},\ x > 0,$$

where $\lambda,\upsilon,a,b > 0$, $n \in \{1,2,\ldots\}$, $p \in (0,1)$ and

$$u_{WWBinom}(x,\lambda,\upsilon,a,b,n,p) \overset{(8)}{=} \frac{npb\upsilon x^{b-1}\exp\left[\left(\frac{x}{a}\right)^{b}-1\right]^{\upsilon}\exp\left[-\left(\frac{\exp\left(x/a\right)^{b}-1}{\lambda}\right)^{\upsilon}\right]\left\{1-p\exp\left[-\frac{\left(x/a\right)^{b}-1}{a}\right]^{\upsilon}\right\}^{n-1}}{\lambda^{\upsilon}a^{b}\left[1-\exp\left(-x/a\right)^{b}\right]\left[1-(1-p)^{n}\right]},$$

with $x > 0$.

5.4 Max Weibull - Weibull Poisson distribution

For this distribution, the *cumulative distribution functions* and the *probability density function* are obtained by the Eq. (9) and Eq. (10), where G is the same as Weibull distribution:

$$U_{WWPoisson}(x,\lambda,\upsilon,a,b,\alpha) \overset{(9)}{=} \frac{\exp\left\{-\alpha\exp\left[-\left(\frac{\exp\left(x/a\right)^{b}-1}{\lambda}\right)^{\upsilon}\right]\right\}-\exp(-\alpha)}{1-\exp(-\alpha)}, x>0$$

and

$$u_{WWPoisson}(x,\lambda,\upsilon,a,b,\alpha) \overset{(10)}{=} \frac{\alpha b\upsilon x^{b-1}\left[\exp\left(x/a\right)^{b}-1\right]^{\upsilon}\exp\left\{-\left(\frac{\exp\left(x/a\right)^{b}-1}{\lambda}\right)^{\upsilon}-\alpha\exp\left[-\frac{\exp\left(x/a\right)^{b}-1}{\lambda}\right]^{\upsilon}\right\}}{\lambda^{\upsilon}a^{b}\left[1-\exp(-\alpha)\right]\left[1-\exp\left(-\left(x/a\right)^{b}\right)\right]},$$

with , where $x>0$, where $\lambda,\upsilon>0$, $a,b,\alpha>0$.

Remark 5.1. In Figures $1\div 4$ Theorem 4.1 is reflected in terms of distribution function and density of probability for both types of Max Weibull-G Binomial distributions, respectively Max Weibull-G Poisson, for different values associated with their parameters, where G is identified with uniform distribution (Figs. 1 and 2), respectively Weibull distribution (Figs. 3 and 4). The parameters for the Max Weibull - Uniform Binomial and the Max Weibull - Uniform Poisson distributions are $\lambda\in\left\{\frac{4}{9},1,\sqrt[3]{4}\right\}$, $\nu\in\{0.5,1,1.5\}$, $\eta=4$, $n=40$, $p=\frac{1}{4}$ and $\alpha=10$ and for the Max Weibull - Weibull Binomial and the Max Weibull - Weibull Poisson distributions the parameters are $\lambda=1$, $\nu\in\{0.5,1,1.5\}$, $a=1.5$, $b=0.5$ $n=30$, $p=\frac{1}{3}$ and $\alpha=10$

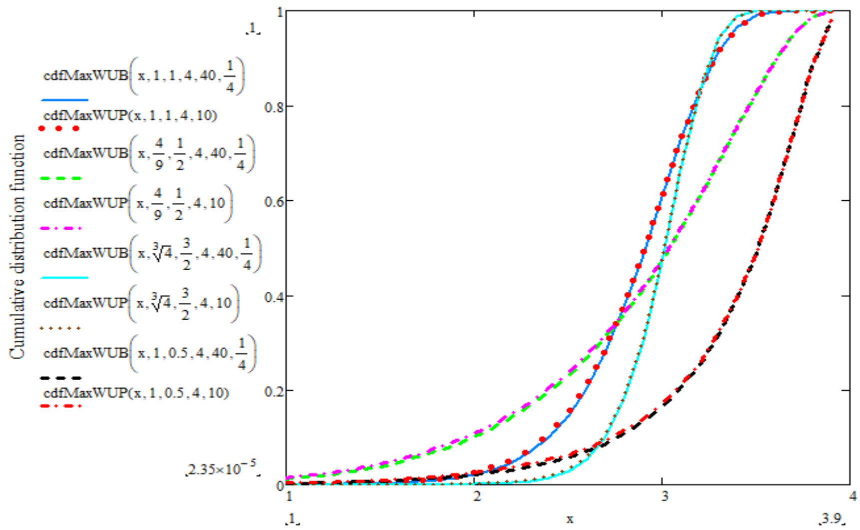

Fig. 1. The Poisson limit theorem in relation with the *cumulative distribution functions* of the Max Weibull - Uniform Binomial distributions ($\mathrm{cdfMaxWUB}(x,\lambda,\upsilon,\phi,n,p)$, $x>0$) and the Max Weibull - Uniform Poisson ($\mathrm{cdfMaxWUP}(x,\lambda,\upsilon,\phi,\alpha)$, $x>0$) , where $G(x,\boldsymbol{\eta})\equiv U(x,\phi)=\frac{x}{\phi},\phi>0,\boldsymbol{\eta}\equiv\phi$

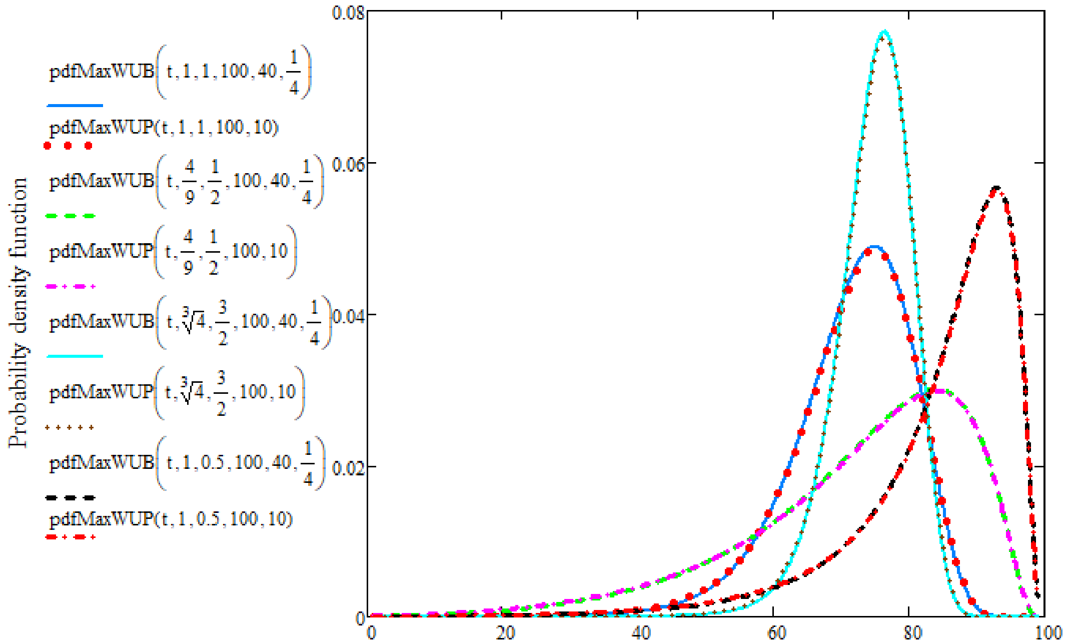

Fig. 2. The Poisson limit theorem in relation with the *probability density function* of the Max Weibull-Uniform Binomial distributions ($\mathrm{pdfMaxWUB}(t,\lambda,\upsilon,\phi,n,p),\ t>0$) and the Max Weibull-Uniform Poisson ($\mathrm{pdfMaxWUP}(t,\lambda,\upsilon,\phi,\alpha),\ t>0$) , where $G(t,\boldsymbol{\eta})\equiv U(t,\phi)=\frac{t}{\phi},\phi>0,\boldsymbol{\eta}\equiv\phi$

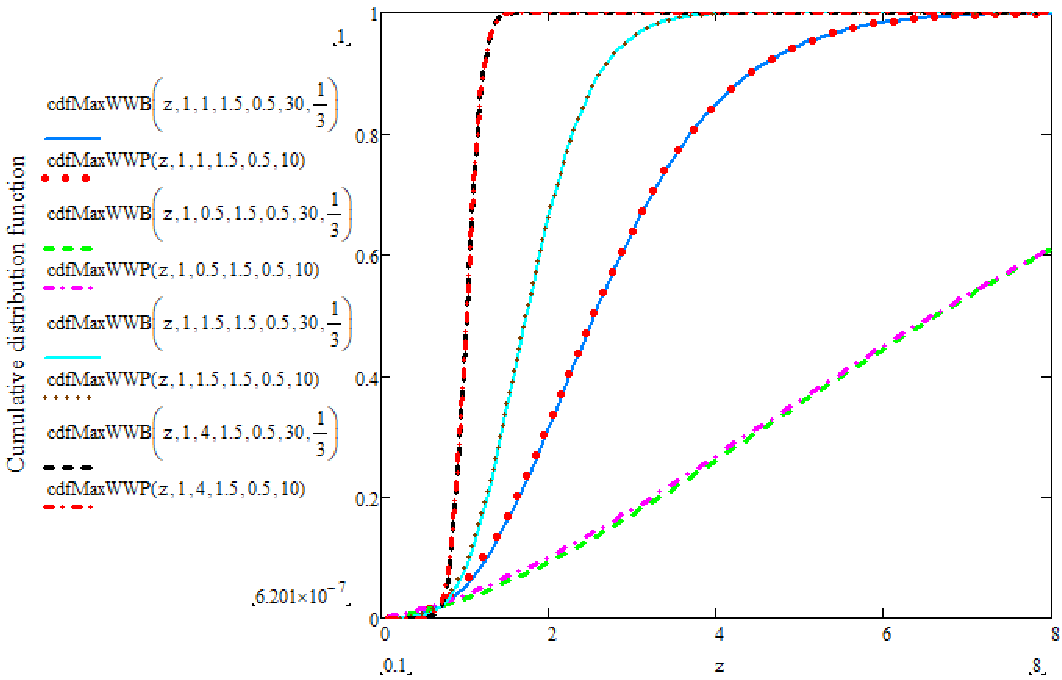

Fig. 3. The Poisson limit theorem in relation with the *cumulative distribution functions* ~~cdf~~ **of the Max Weibull - Weibull Binomial distributions (** $\mathrm{cdfMaxWWB}(z,\lambda,\upsilon,a,b,n,p),\ z>0$ **) and the Max Weibull -**

Weibull Poisson $cdfMaxWWP(z, \lambda, \upsilon, a, b, \alpha),\ z > 0$**, where**

$$G(z, \boldsymbol{\eta}) \equiv Weibull(z, a, b) = 1 - \exp\left[-\left(\frac{z}{a}\right)^b\right], \quad a, b > 0, \boldsymbol{\eta} \equiv (a, b)$$

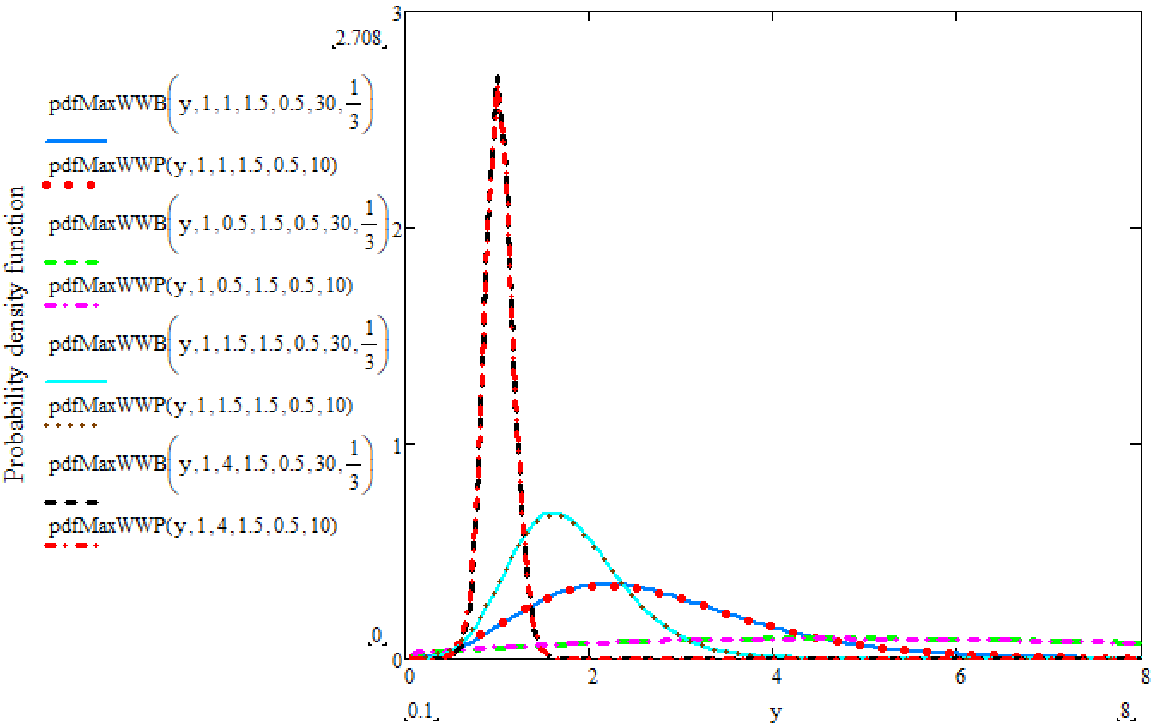

Fig. 4. The Poisson limit theorem in relation with the *probability density function* ~~*pdf*~~ **of the Max Weibull - Weibull Binomial distributions** $pdfMaxWWB(y, \lambda, \upsilon, a, b, n, p),\ y > 0$ **and the Max Weibull - Weibull Poisson** $pdfMaxWWP(y, \lambda, \upsilon, a, b, \alpha),\ y > 0$**, where**

$$G(y, \boldsymbol{\eta}) \equiv Weibull(y, a, b) = 1 - \exp\left[-\left(\frac{y}{a}\right)^b\right], \quad a, b > 0, \boldsymbol{\eta} \equiv (a, b)$$

6 Conclusion

In this study, a new class of power series distributions called Max Weibull-G has been analyzed. If the lifetime (Y) of a system has a Max Weibull-G (U_{WG}) power series distribution, then the chance for a component with the maximum lifetime Y to degrade at time t is $U_{WG}(t, \lambda, \upsilon, \eta)/(1 - U_{WG}(t, \lambda, \upsilon, \eta))$. Thus, the moment when a system enters the state of degradation can be determined if we know the distribution of the maximum lifetime from all the other lifetimes that characterize the system.

The Poisson Limit Theorem has been formulated for situations when the random variables number of sum is a zero truncated binomial distribution and limit distribution the Poisson type distribution. The theorems are shown under the graphical representation in Figures 1, 2, 3 and 4.

The validation of the Max Weibull-G power series distributions through the EM algorithm is a new research direction.

Competing Interests

Author has declared that no competing interests exist.

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