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Free-Form Laminated Doubly-Curved Shells and Panels of Revolution Resting on Winkler-Pasternak Elastic Foundations: A 2-D GDQ Solution for Static and Free Vibration Analysis

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ABSTRACT

This work presents the static and dynamic analyses of laminated doubly-curved shells and panels of revolution resting on Winkler-Pasternak elastic foundations using the Generalized Differential Quadrature (GDQ) method. The analyses are worked out considering the First-order Shear Deformation Theory (FSDT) for the above mentioned moderately thick structural elements. The effect of the shell curvatures is included from the beginning of the theory formulation in the kinematic model. The solutions are given in terms of generalized displacement components of points lying on the middle surface of the shell. Simple Rational Bézier curves are used to define the meridian curve of the revolution structures. The discretization of the system by means of the GDQ technique leads to a standard linear problem for the static analysis and to a standard linear eigenvalue problem for the dynamic analysis. Comparisons between the present formulation and the Reissner-Mindlin theory are presented. Furthermore, GDQ results are compared with those obtained by using commercial programs. Very good agreement is observed. Finally, new results are presented in order to investigate the effects of the Winkler modulus, the Pasternak modulus and the inertia of the elastic foundation on the behavior of laminated shells of revolution.

Keywords: Doubly-Curved Shells of Revolution; Rational Bézier Curves; Laminated Composite Shells; Winkler-Pasternak Foundation; First-Order Shear Deformation Theory; Generalized Differential Quadrature Method

1. Introduction

During the last sixty years, two-dimensional linear theories of thin shells have been developed including important contributions by Timoshenko and Woinowsky-Krieger [1], Flügge [2], Gol'denveizer [3], Novozhilov [4], Vlasov [5], Ambartusumyan [6], Kraus [7], Leissa [8,9], Markuš [10], Ventsel and Krauthammer [11] and Soedel [12]. All these contributions are based on the Kirchhoff-Love assumptions. The transverse shear deformation has been incorporated into shell theories by following the theory of Reissner-Mindlin [13], also named First-order Shear Deformation Theory (FSDT). Abandoning the assumption related to the preservation of the normals to the shell middle surface after the deformation, a comprehensive analysis for elastic isotropic shells was made by Kraus [7], Gould [14,15] and Qatu [16,17]. The present work is just based on the FSDT. In order to include the effect of the initial curvature in the evaluation of the stress resultants a generalization of the Reissner-Mindlin (RM) theory has been proposed in literature by Kraus [7], Qatu [16,17] and Toorani and Lakis [18,19]. There are three different ways to evaluate the engineering elastic constants in the study of curved shells. The first is the Reissner-Mindlin approach [7] that consists in neglecting the effect of curvatures. Using this approach the engineering elastic stiffnesses are constant and do not depend on curvatures. The second one, proposed by Kraus [7] and Toorani and Lakis [18], is based on the Taylor expansion, while the third one proposed by Qatu [16] consists in the exact integration of the elastic constants. As a consequence of the use of these considerations, the stress resultants directly depend on the geometry of the structure in terms of the curvature coefficients. In this latter case, the hypothesis of the symmetry of the in-plane shearing force resultants and the torsional couples de-

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clines. A further improvement of the previous theories of shells has been proposed by Toorani and Lakis [19]. In the present work their kinematic model is used in order to include the effect of the curvature from the beginning of the shell formulation. In this way, the strain relationships have to change and, as a consequence, the equilibrium equations in terms of displacements have to be modified. In the present paper, the proposed shell theory, named General Shell Theory (GST), is considered and compared with the Reissner-Mindlin (RM) theory. Comparisons between these two different formulations are presented in this paper. Several studies dealing with the shells theory have been presented years before. The most popular numerical tool used to perform the static and dynamic analyses is currently the finite element method [14,15,20]. The generalized collocation method based on the ring element method has also been applied. In this method, each static and kinematic variable is transformed into a theoretically infinite Fourier series of harmonic components, with respect to the circumferential co-ordinate [21,22]. In other words, when dealing with a completely closed shell, the 2D problem can be reduced using standard Fourier decomposition. For a panel, however, it is not possible to perform such a reduction operation, and the two dimensional field must be directly dealt, as it will just be done in the present work. Furthermore, the system of second-order linear partial differential equations is solved, without resorting to the one-dimensional formulation of the equilibrium of the shell. Complete revolution shells are obtained as special cases of shell panels by satisfying the kinematical and physical compatibility at the common meridian with $\vartheta = 0, 2\pi$. The excellent mathematical and computational algorithmic properties, combined with successful industrial applications, have contributed to the enormous popularity of the Rational Bézier and Non-Uniform Rational B-Splines (NURBS) curves [23-25]. These curves allow to generalize the shape of the shell meridian and can be used for the optimization of the structure itself. By introducing the Differential Quadrature rule [26] and the simple mathematical formulation of the Rational Bézier and NURBS curves [23-25], it is possible to numerically evaluate the geometric parameters of a free-form shell of revolution. For the sake of simplicity and without loosing generality, only Rational Bézier curves are used in this study. Due to the increasing importance of the interaction of shells with the elastic medium, the Winkler-Pasternak foundation is introduced. Differently from papers presented in literature [27-31], all the effects of the foundation, except the damping, are separately considered. New results are presented in order to investigate the effects of the Winkler modulus, the Pasternak modulus and the inertia of the elastic foundation on the behavior of laminated shells of revolution. The mathematical fundamentals and recent developments of the GDQ method as well as its major applications in engineering are discussed in detail in the book by Shu [26]. The interest of researches in this procedure is increasing due to its great simplicity and versatility. As shown in the literature [32], GDQ technique is a global method which can obtain very accurate numerical results by using a considerably small number of grid points. Therefore, this simple direct procedure has been applied in a large number of cases [33-85] to circumvent the difficulties of programming complex algorithms for the computer, as well as to reduce the computational time. In conclusion, the aim of the present paper is to demonstrate an efficient and accurate application of the Differential Quadrature approach, by solving the equations governing the static and the free vibration of laminated composite doubly-curved moderately thick shells and panels of revolution. Summarizing, this research deals with four aspects. The first is the improvement of the Reissner-Mindlin Theory using a different kinematical model. In this way the effect of the curvature of the shell structure is considered from the beginning of the theory derivation. The second is the generalization of the shape of the shell meridian. The Differential Quadrature rule is used to evaluate the geometric parameters needed to describe the geometry of the structure when a Rational Bézier meridian curve is assumed. The third is the investigation of the effects of Winkler-Pasternak foundations on the behavior of the shell structures in static and dynamic analyses. All the effects of the foundation are separately considered. The fourth is the use of the Generalized Differential Quadrature method to solve the governing shell equations.

2. Shell Fundamental Equations

The basic configuration of the problem herein considered is a laminated composite doubly-curved shell [83] as shown in **Figure 1**. The co-ordinates along the meridian and circumferential directions of the reference surface are φ and s, respectively. The distance of each point from the shell mid-surface along the normal is ζ. It is considered a laminated composite shell made of l laminae or plies, where the total thickness of the shell h is defined as:

$$h = \sum_{k=1}^{l} h_k \tag{1}$$

in which $h_k = \zeta_{k+1} - \zeta_k$ is the thickness of the *k*-th lamina or ply. In this work, doubly-curved shells of revolution are considered. For this type of structures the analytical expressions of the meridian curve are reported in the work by Tornabene [78], so that no further considerations will be introduced. The angle formed by the extended normal $\boldsymbol{n}$ to the reference surface and the axis of rotation x_3, or the geometric axis x_3' of the meridian curve, is defined as the meridian angle φ; the angle between the radius of the parallel circle $R_0(\varphi)$ and the x_1 axis is designated as the circumferential angle ϑ, as shown in **Figure 2**.

For these structures the parametric co-ordinates (φ, s) define, respectively, the meridian curves and the parallel circles upon the middle surface of the shell. The curvilinear abscissa $s(\varphi)$ of a generic parallel is related to the circumferential angle ϑ by the relation $s = \vartheta R_0$. The horizontal radius $R_0(\varphi)$ of a generic parallel of the shell represents the distance of each point from the axis of revolution x_3. R_b is the shift of the geometric axis of the curved meridian x_3' with reference to the axis of revolution x_3. The position of an arbitrary point within the shell material is defined by co-ordinates $\varphi(\varphi_0 \le \varphi \le \varphi_1)$, $s(0 \le s \le s_0)$ upon the middle surface, and ζ directed along the outward normal and measured from the reference surface $(-h/2 \le \zeta \le h/2)$. The geometry of shells considered [83] is a surface of revolution (**Figure 2**).

A simple way to define a general meridian curve is to use the well-known Rational Bézier representation of a plane curve [24,25,80]. In particular, it is possible to describe a Rational Bézier curve in the following manner:

$$\begin{aligned} \hat{x}_1(u) &= \sum_{i=0}^{n} B_{i,n}(u) w_i \bar{x}_{1i} \Big/ \sum_{i=0}^{n} B_{i,n}(u) w_i \\ \hat{x}_3'(u) &= \sum_{i=0}^{n} B_{i,n}(u) w_i \bar{x}_{3i}' \Big/ \sum_{i=0}^{n} B_{i,n}(u) w_i \end{aligned} \tag{2}$$

where $u \in [0,1]$ is the curve parameter, w_i are the weight coefficients and $(\bar{x}_{1i}, \bar{x}_{3i}')$ are the co-ordinates of the curve control points. Furthermore, the classical n-th degree Bernstein polynomial formulations are given by:

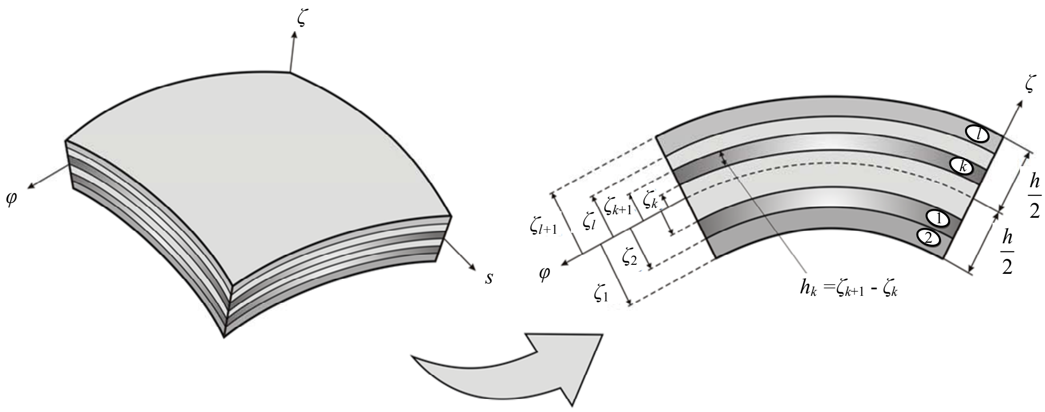

Figure 1. Co-ordinate system of a laminated composite doubly-curved shell.

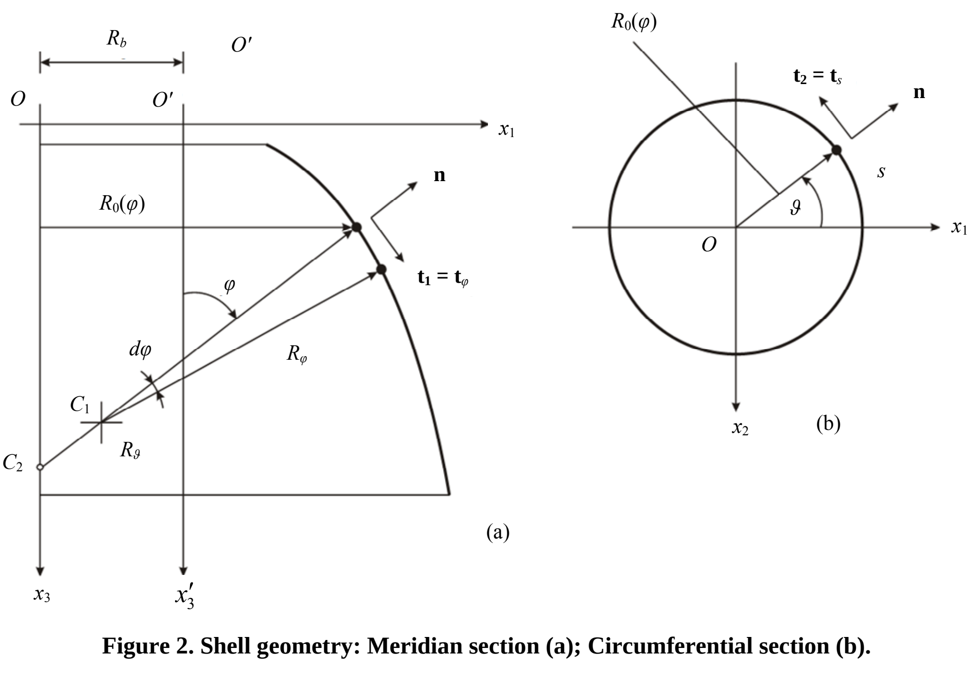

Figure 2. Shell geometry: Meridian section (a); Circumferential section (b).

$$B_{i,n}(u)=\frac{n!}{i!(n-i)!}u^i(1-u)^{n-i} \tag{3}$$

In this way, only the co-ordinates of the curve $(\hat{x}_{1i},\hat{x}'_{3i}), i=1,2,\cdots,m$, are known in the co-ordinate reference system $O'x_1x'_3$. In order to solve the shell problem, it is important to express the horizontal radius $R_0(\varphi)$ of a generic parallel and the radii of curvature $R_\varphi(\varphi), R_s(\varphi)$ in the meridian and circumferential directions as functions of φ. Based on the differential geometry [7,12,57,80,83], the radius of curvature of the meridian curve can be described as a function of x'_3 using the following expression:

$$R_\varphi(x'_3)=\left(1+\left(\frac{\mathrm{d}x_1}{\mathrm{d}x'_3}\right)^2\right)^{\frac{3}{2}}\Bigg/\left|\frac{\mathrm{d}^2x_1}{\mathrm{d}x'^2_3}\right| \tag{4}$$

It is worth noting that the derivatives of the meridian curve are not known a priori, so that a numeric method to evaluate the first and second derivatives of the meridian curve is required. The differential quadrature rule allows to approximate these derivatives using the following definition [26]:

$$\left.\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right|_{x=x_i}=\sum_{j=1}^{N}\varsigma_{ij}^{(n)}f(x_j), \text{for } i=1,2,\cdots,N \tag{5}$$

where $\varsigma_{ij}^{(n)}$ are the weighting coefficients of the n-th order derivative. By discretizing the domain

$$I\in[\hat{x}'_{31},\hat{x}'_{3m}]$$

using the Chebyshev-Gauss-Lobatto (C-G-L) grid distribution:

$$\hat{x}'_{3i}=\left(1-\cos\left(\frac{i-1}{N-1}\pi\right)\right)\frac{(\hat{x}'_{3m}-\hat{x}'_{31})}{2}+\hat{x}'_{31}, \\ \text{for } i=1,2,\cdots,N, \text{for } \hat{x}'_3\in[\hat{x}'_{31},\hat{x}'_{3m}] \tag{6}$$

and interpolating the $\hat{x}_1$ co-ordinates of the curve points derived by the Equations (2) using the previous calculated points (6), the general curve can be represented by the new co-ordinates points $(\hat{x}_{1i},\hat{x}'_{3i})$, for $i=1,2,\cdots,N$. Applying the differential quadrature definition (5), the expression (4) assumes the following discrete aspect:

$$R_\varphi(\hat{x}'_{3i})=\left(1+\left(\sum_{j=1}^{N}\varsigma_{ij}^{\hat{x}'_3(1)}\hat{x}_{1i}\right)^2\right)^{\frac{3}{2}}\Bigg/\left|\sum_{j=1}^{N}\varsigma_{ij}^{\hat{x}'_3(2)}\hat{x}_{1i}\right|, \\ \text{for } i=1,2,\cdots,N \tag{6}$$

where $\varsigma_{ij}^{\hat{x}'_3(n)}$ are the weighting coefficients evaluated in the domain $I\in[\hat{x}'_{31},\hat{x}'_{3m}]$. As a results of the differential geometry [7,12,57,80,83], it is possible to introduce the following relation:

$$\varphi=\frac{\pi}{2}-\arctan\left(\frac{\mathrm{d}x_1}{\mathrm{d}x'_3}\right) \tag{7}$$

By using the differential quadrature definition (5), the relation (7) can be expressed in the discrete form:

$$\hat{\varphi}_i=\hat{\varphi}(\hat{x}'_{3i})=\frac{\pi}{2}-\arctan\left(\sum_{j=1}^{N}\varsigma_{ij}^{\hat{x}'_3(1)}\hat{x}_{1i}\right), \\ \text{for } i=1,2,\cdots,N \tag{8}$$

By discretizing the domain $I_\varphi\in[\hat{\varphi}_1,\hat{\varphi}_N]$ using the Chebyshev-Gauss-Lobatto (C-G-L) grid distribution:

$$\varphi_i=\left(1-\cos\left(\frac{i-1}{N-1}\pi\right)\right)\frac{(\hat{\varphi}_N-\hat{\varphi}_1)}{2}+\hat{\varphi}_1, \\ \text{for } i=1,2,\cdots,N, \text{for } \varphi\in[\hat{\varphi}_1,\hat{\varphi}_N] \tag{9}$$

and interpolating the $\hat{x}_1$ and $\hat{x}'_3$ co-ordinates of the curve points using the calculated points (9), the general curve can be represented by the following new co-ordinate points $(\tilde{x}_{1i},\tilde{x}'_{3i})$, for $i=1,2,\cdots,N$. Thus, all the discrete points of the curve are determined in terms of the co-ordinates $(\tilde{x}_{1i},\tilde{x}'_{3i})$ and the angle φ_i. In the **Figure 3**, a Rational Bézier curve, its control points and the curve co-ordinates $(\tilde{x}_{1i},\tilde{x}'_{3i})$, evaluated as above exposed, are represented [80]. The vectors of the control points and the weights used in **Figure 3** are the following:

$$\bar{\boldsymbol{x}}_1=[0.2\quad 0.7\quad 1.2\quad 1.4\quad 1.4\quad 1.2] \\ \bar{\boldsymbol{x}}'_3=[0\quad 0.2\quad 0.6\quad 1\quad 1.5\quad 2] \\ \boldsymbol{w}=[1\ 1\ 1\ 1\ 1\ 1] \tag{10}$$

Based on the previous considerations, the horizontal radius $R_0(\varphi)$ of a shell of revolution assumes the following discrete form:

$$R_{0i}=R_0(\varphi_i)=\tilde{x}_{1i}+R_b, \text{for } i=1,2,\cdots,N \tag{11}$$

For doubly-curved revolution shells the Gauss-Codazzi relation can be expressed as follows:

$$\frac{\mathrm{d}R_0}{\mathrm{d}\varphi}=R_\varphi\cos\varphi \tag{12}$$

By using the differential quadrature definition (5), it is possible to determine the radius of curvature $R_\varphi(\varphi)$ in meridian direction and its first and second derivatives in discrete form:

$$R_{\varphi i}=R_\varphi(\varphi_i)=\frac{1}{\cos\varphi_i}\sum_{j=1}^{N}\varsigma_{ij}^{\varphi(1)}R_{0i}, \text{for } i=1,2,\cdots,N \tag{13}$$

$$\left.\frac{\mathrm{d}R_\varphi}{\mathrm{d}\varphi}\right|_i=\left.\frac{\mathrm{d}R_\varphi}{\mathrm{d}\varphi}\right|_{\varphi_i}=\sum_{j=1}^{N}\varsigma_{ij}^{\varphi(1)}R_{\varphi i}, \text{for } i=1,2,\cdots,N \tag{14}$$

$$\left.\frac{\mathrm{d}^2R_\varphi}{\mathrm{d}\varphi^2}\right|_i=\left.\frac{\mathrm{d}^2R_\varphi}{\mathrm{d}\varphi^2}\right|_{\varphi_i}=\sum_{j=1}^{N}\varsigma_{ij}^{\varphi(2)}R_{\varphi i}, \text{for } i=1,2,\cdots,N \tag{15}$$

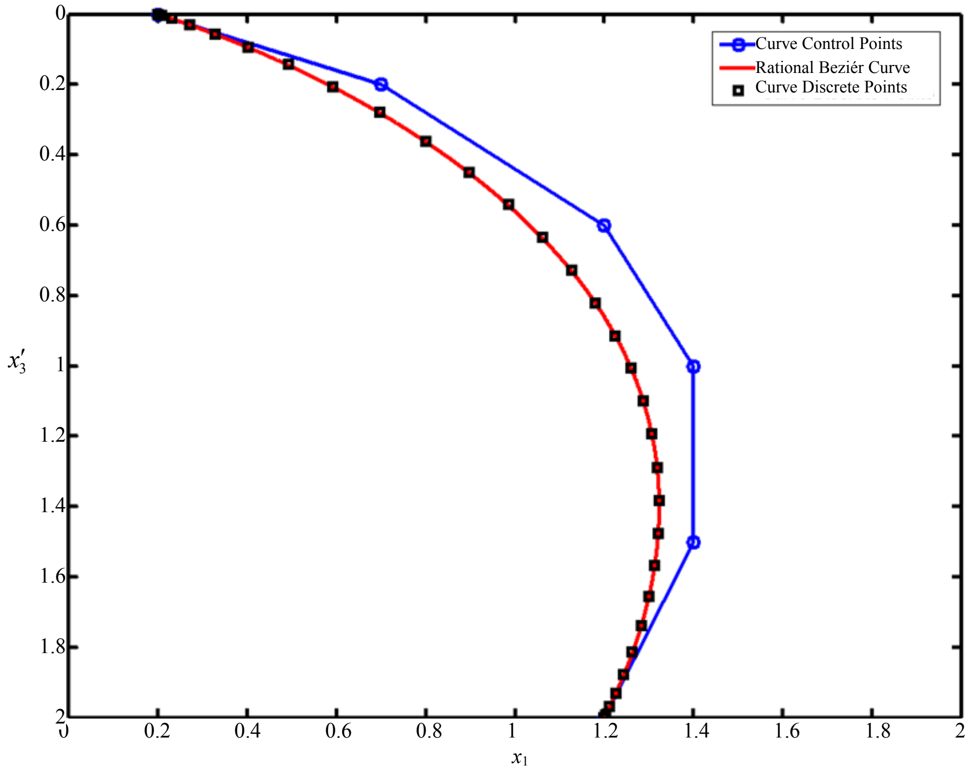

Figure 3. A Rational Bézier curve $(\hat{x}_{1i}, \hat{x}'_{3i})$, its control points $(\hat{x}_{1i}, \hat{x}'_{3i})$ and curve evaluated discrete points $(\hat{x}_{1i}, \hat{x}'_{3i})$.

Finally, as a results of the differential geometry [7,12, 57,80,83], the radius of curvature $R_s(\varphi)$ in circumferential direction for a shell of revolution can be expressed in a discrete form as follows:

$$R_{si} = R_s(\varphi_i) = \frac{R_{0i}}{\sin\varphi_i}, \text{ for } i = 1,2,\cdots,N \qquad (16)$$

It is worth noting that, following the previous considerations, all the useful geometric parameters describing the surface of revolution under consideration are known in discrete form (11)-(16). As shown, the differential quadrature rule (5) has been used to approximate the derivatives needed for the definition of the geometry of a shell of revolution. As concerns the shell theory, the present work is based on the following assumptions: 1) the transverse normal is inextensible so that the normal strain is equal to zero: $\varepsilon_n = \varepsilon_n(\varphi, s, \zeta, t) = 0$; 2) the transverse shear deformation is considered to influence the governing equations so that normal lines to the reference surface of the shell before deformation remain straight, but not necessarily normal after deformation (a relaxed Kirchhoff-Love hypothesis); 3) the shell deflections are small and the strains are infinitesimal; 4) the shell is moderately thick, therefore it is possible to assume that the thickness direction normal stress is negligible so that the in-plane assumption can be invoked: $\sigma_n = \sigma_n(\varphi, s, \zeta, t) = 0$; 5) the linear elastic behavior of anisotropic materials is assumed; 6) the rotary inertia and the initial curvature are also taken into account. Consistent with the assumptions of a moderately thick shell theory reported above, the displacement field considered in this study follows the First-order Shear Deformation Theory and it can be put in the following form:

$$\begin{aligned} U_\varphi(\varphi, s, \zeta, t) &= \left(1 + \frac{\zeta}{R_\varphi}\right) u_\varphi(\varphi, s, t) + \zeta\beta_\varphi(\varphi, s, t) \\ U_s(\varphi, s, \zeta, t) &= \left(1 + \frac{\zeta}{R_s}\right) u_s(\varphi, s, t) + \zeta\beta_s(\varphi, s, t) \\ W(\varphi, s, \zeta, t) &= w(\varphi, s, t) \end{aligned} \qquad (17)$$

where u_φ, u_s, w are the displacement components of points lying on the middle surface $(\zeta = 0)$ of the shell, along meridian, circumferential and normal directions, respectively, while t is the time variable. β_φ and β_s are normal-to-mid-surface rotations, respectively. The kinematic hypothesis expressed by Equations (17) should be supplemented by the statement that the shell deflections are small and strains are infinitesimal, that is $w(\varphi, s, t) \ll h$. The in-plane displacements U_φ and U_s vary linearly through the thickness, while W remains independent of ζ. It should be also remarked that, differently from the previous works by Tornabene [65,77,78], the displacement field has been improved taking into account the real geometry of the shell and in particular the curvature effect has been directly introduced into the kinematical model as proposed by Toorani and Lakis [19]. Due to the change of the kinematical model the relationships between strains and displacements along the shell reference surface $(\zeta = 0)$ become the following:

$$\begin{bmatrix} \varepsilon_\varphi^0 \\ \varepsilon_s^0 \\ \gamma_\varphi^0 \\ \gamma_s^0 \\ \chi_\varphi^0 \\ \chi_s^0 \\ \omega_\varphi^0 \\ \omega_s^0 \\ \gamma_{\varphi n}^0 \\ \gamma_{sn}^0 \end{bmatrix} = \begin{bmatrix} \frac{1}{R_\varphi}\frac{\partial}{\partial\varphi} & 0 & \frac{1}{R_\varphi} & 0 & 0 \\ \frac{\cos\varphi}{R_0} & \frac{\partial}{\partial s} & \frac{\sin\varphi}{R_0} & 0 & 0 \\ 0 & \frac{1}{R_\varphi}\frac{\partial}{\partial\varphi} & 0 & 0 & 0 \\ \frac{\partial}{\partial s} & -\frac{\cos\varphi}{R_0} & 0 & 0 & 0 \\ \frac{1}{R_\varphi^2}\frac{\partial}{\partial\varphi} - \frac{1}{R_\varphi^3}\frac{\partial R_\varphi}{\partial\varphi} & 0 & 0 & \frac{1}{R_\varphi}\frac{\partial}{\partial\varphi} & 0 \\ \frac{\cos\varphi}{R_0 R_\varphi} & \frac{\sin\varphi}{R_0}\frac{\partial}{\partial s} & 0 & \frac{\cos\varphi}{R_0} & \frac{\partial}{\partial s} \\ 0 & \frac{\sin\varphi}{R_\varphi R_0}\frac{\partial}{\partial\varphi} + \frac{\cos\varphi}{R_\varphi R_0} - \frac{\sin\varphi\cos\varphi}{R_0^2} & 0 & 0 & \frac{1}{R_\varphi}\frac{\partial}{\partial\varphi} \\ \frac{1}{R_\varphi}\frac{\partial}{\partial s} & -\frac{\sin\varphi\cos\varphi}{R_0^2} & 0 & \frac{\partial}{\partial s} & -\frac{\cos\varphi}{R_0} \\ 0 & 0 & \frac{1}{R_\varphi}\frac{\partial}{\partial\varphi} & 1 & 0 \\ 0 & 0 & \frac{\partial}{\partial s} & 0 & 1 \end{bmatrix} \begin{bmatrix} u_\varphi \\ u_s \\ w \\ \beta_\varphi \\ \beta_s \end{bmatrix} \tag{18}$$

that are different from those presented in previous papers [77,78]. In the above Equations (19), the first four strains $\varepsilon_\varphi^0, \varepsilon_s^0, \gamma_\varphi^0, \gamma_s^0$ are the in-plane meridian, circumferential and shearing components, and $\chi_\varphi^0, \chi_s^0, \omega_\varphi^0, \omega_s^0$ are the analogous curvature changes. The last two components $\gamma_{\varphi n}^0, \gamma_{sn}^0$ are the transverse shearing strains. The shell composition assumed in the following is a laminated composite linear elastic material. Accordingly, the following constitutive equations relate internal stress resultants and internal couples with generalized strain components (18) on the middle surface:

$$\begin{bmatrix} N_\varphi \\ N_s \\ N_{\varphi s} \\ N_{s\varphi} \\ M_\varphi \\ M_s \\ M_{\varphi s} \\ M_{s\varphi} \\ T_\varphi \\ T_s \end{bmatrix} = \begin{bmatrix} \overline{A}_{11\varphi}^{(0)} & A_{12}^{(0)} & \overline{A}_{16\varphi}^{(0)} & A_{16}^{(0)} & \overline{A}_{11\varphi}^{(1)} & A_{12}^{(1)} & \overline{A}_{16\varphi}^{(1)} & A_{16}^{(1)} & 0 & 0 \\ A_{12}^{(0)} & \overline{A}_{22s}^{(0)} & A_{26}^{(0)} & \overline{A}_{26s}^{(0)} & A_{12}^{(1)} & \overline{A}_{22s}^{(1)} & A_{26}^{(1)} & \overline{A}_{26s}^{(1)} & 0 & 0 \\ \overline{A}_{16\varphi}^{(0)} & A_{26}^{(0)} & \overline{A}_{66\varphi}^{(0)} & A_{66}^{(0)} & \overline{A}_{16\varphi}^{(1)} & A_{26}^{(1)} & \overline{A}_{66\varphi}^{(1)} & A_{66}^{(1)} & 0 & 0 \\ A_{16}^{(0)} & \overline{A}_{26s}^{(0)} & A_{66}^{(0)} & \overline{A}_{66s}^{(0)} & A_{16}^{(1)} & \overline{A}_{26s}^{(1)} & A_{66}^{(1)} & \overline{A}_{66s}^{(1)} & 0 & 0 \\ \overline{A}_{11\varphi}^{(1)} & A_{12}^{(1)} & \overline{A}_{16\varphi}^{(1)} & A_{16}^{(1)} & \overline{A}_{11\varphi}^{(2)} & A_{12}^{(2)} & \overline{A}_{16\varphi}^{(2)} & A_{16}^{(2)} & 0 & 0 \\ A_{12}^{(1)} & \overline{A}_{22s}^{(1)} & A_{26}^{(1)} & \overline{A}_{26s}^{(1)} & A_{12}^{(2)} & \overline{A}_{22s}^{(2)} & A_{26}^{(2)} & \overline{A}_{26s}^{(2)} & 0 & 0 \\ \overline{A}_{16\varphi}^{(1)} & A_{26}^{(1)} & \overline{A}_{66\varphi}^{(1)} & A_{66}^{(1)} & \overline{A}_{16\varphi}^{(2)} & A_{26}^{(2)} & \overline{A}_{66\varphi}^{(2)} & A_{66}^{(2)} & 0 & 0 \\ A_{16}^{(1)} & \overline{A}_{26s}^{(1)} & A_{66}^{(1)} & \overline{A}_{66s}^{(1)} & A_{16}^{(2)} & \overline{A}_{26s}^{(2)} & A_{66}^{(2)} & \overline{A}_{66s}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa\overline{A}_{44\varphi}^{(0)} & \kappa A_{45}^{(0)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa A_{45}^{(0)} & \kappa\overline{A}_{55s}^{(0)} \end{bmatrix} \begin{bmatrix} \varepsilon_\varphi^0 \\ \varepsilon_s^0 \\ \gamma_\varphi^0 \\ \gamma_s^0 \\ \chi_\varphi^0 \\ \chi_s^0 \\ \omega_\varphi^0 \\ \omega_s^0 \\ \gamma_{\varphi n}^0 \\ \gamma_{sn}^0 \end{bmatrix} \tag{19}$$

where the elastic engineering stiffnesses $\overline{A}_{ijm}^{(q)}$ which depend on curvatures are defined as follows (see Appendix for more details):

$$\bar{A}_{ijm}^{(q)} \cong A_{ij}^{(q)} + \left(\frac{1}{R_s} - \frac{1}{R_\varphi}\right)\sum_{p=1}^{r}(-1)^{p+n_m}\frac{A_{ij}^{(q+p)}}{R_m^{p-1}} = \sum_{k=1}^{l}\left(\int_{\zeta_k}^{\zeta_{k+1}} \bar{Q}_{ij}^{(k)}\zeta^q \mathrm{d}\zeta + \left(\frac{1}{R_s} - \frac{1}{R_\varphi}\right)\sum_{p=1}^{r}\frac{(-1)^{p+n_m}}{R_m^{p-1}}\int_{\zeta_k}^{\zeta_{k+1}} \bar{Q}_{ij}^{(k)}\zeta^{q+p}\mathrm{d}\zeta\right) \tag{20}$$
$$m = \varphi, s, n_\varphi = 1, n_s = 2, q = 0,1,2, r = 1,2,3,\cdots,20$$

Several approaches can be found in literature to evaluate the engineering elastic constants $\bar{A}_{ijm}^{(q)}$ [7,16-18]. It is worth noting that due to the fact that the elastic engineering stiffnesses $\bar{A}_{ijm}^{(q)}$ depend on curvatures, the corresponding derivatives respect to the co-ordinates along the meridian φ and circumferential s directions of the reference surface have to be evaluated. In order to perform this operation, the Differential Quadrature rule [26] is used. Thus, the derivatives of the elastic engineering stiffnesses $\bar{A}_{ijm}^{(q)}$ are numerically evaluated. κ is the shear correction factor, which is usually taken equal to $\kappa = 5/6$, such as in the present work. In particular, the determination of shear correction factors for composite laminated structures is still an unresolved issue, because these factors depend on various parameters [18]. In Equations (19), the four components $N_\varphi, N_s, N_{\varphi s}, N_{s\varphi}$ are the in-plane meridian, circumferential and shearing force resultants, and $M_\varphi, M_s, M_{\varphi s}, M_{s\varphi}$ are the analogous couples, while T_φ, T_s are the transverse shear force resultants. In the above definitions (19) the symmetry of shearing force resultants $N_{\varphi s}, N_{s\varphi}$ and torsional couples $M_{\varphi s}, M_{s\varphi}$ is not assumed as a further hypothesis, as done in Reissner-Mindlin theory. This hypothesis is in fact satisfied only in the case of spherical shells and flat plates. The assumption under discussion is derived from the consideration that ratios $\zeta/R_\varphi, \zeta/R_s$ cannot be neglected with respect to unity. For the *k*-th orthotropic lamina the elastic constants $\bar{Q}_{ij}^{(k)}$ in the laminate co-ordinate system $O'\varphi s\zeta$ can be written as:

$$\begin{aligned}
\bar{Q}_{11}^{(k)} &= Q_{11}^{(k)}\cos^4\theta^{(k)} + 2\left(Q_{12}^{(k)} + 2Q_{66}^{(k)}\right)\sin^2\theta^{(k)}\cos^2\theta^{(k)} + Q_{22}^{(k)}\cos^4\theta^{(k)} \\
\bar{Q}_{12}^{(k)} &= \left(Q_{11}^{(k)} + Q_{22}^{(k)} - 4Q_{66}^{(k)}\right)\sin^2\theta^{(k)}\cos^2\theta^{(k)} + Q_{12}^{(k)}\left(\sin^4\theta^{(k)} + \cos^4\theta^{(k)}\right) \\
\bar{Q}_{22}^{(k)} &= Q_{11}^{(k)}\sin^4\theta^{(k)} + 2\left(Q_{12}^{(k)} + 2Q_{66}^{(k)}\right)\sin^2\theta^{(k)}\cos^2\theta^{(k)} + Q_{22}^{(k)}\cos^4\theta^{(k)} \\
\bar{Q}_{16}^{(k)} &= \left(Q_{11}^{(k)} - Q_{12}^{(k)} - 2Q_{66}^{(k)}\right)\sin\theta^{(k)}\cos^3\theta^{(k)} + \left(Q_{12}^{(k)} - Q_{22}^{(k)} + 2Q_{66}^{(k)}\right)\sin^3\theta^{(k)}\cos\theta^{(k)} \\
\bar{Q}_{26}^{(k)} &= \left(Q_{11}^{(k)} - Q_{12}^{(k)} - 2Q_{66}^{(k)}\right)\sin^3\theta^{(k)}\cos\theta^{(k)} + \left(Q_{12}^{(k)} - Q_{22}^{(k)} + 2Q_{66}^{(k)}\right)\sin\theta^{(k)}\cos^3\theta^{(k)} \\
\bar{Q}_{66}^{(k)} &= \left(Q_{11}^{(k)} + Q_{22}^{(k)} - 2Q_{12}^{(k)} - 2Q_{66}^{(k)}\right)\sin^2\theta^{(k)}\cos^2\theta^{(k)} + Q_{66}^{(k)}\left(\sin^4\theta^{(k)} + \cos^4\theta^{(k)}\right) \\
\bar{Q}_{44}^{(k)} &= Q_{44}^{(k)}\cos^2\theta^{(k)} + Q_{55}^{(k)}\sin^2\theta^{(k)} \\
\bar{Q}_{45}^{(k)} &= \left(Q_{44}^{(k)} - Q_{55}^{(k)}\right)\cos\theta^{(k)}\sin\theta^{(k)} \\
\bar{Q}_{55}^{(k)} &= Q_{55}^{(k)}\cos^2\theta^{(k)} + Q_{44}^{(k)}\sin^2\theta^{(k)}
\end{aligned} \tag{21}$$

where $\theta^{(k)}$ is the orientation angle of the principal material co-ordinate system $O'\hat{\varphi}\hat{s}\hat{\zeta}$ of the *k*-th orthotropic ply with respect to the laminate co-ordinate system $O'\varphi s\zeta$. Furthermore, the elastic constants $Q_{ij}^{(k)}$ in the material co-ordinate system $O'\hat{\varphi}\hat{s}\hat{\zeta}$ are expressed as follows:

$$Q_{11}^{(k)} = \frac{E_1^{(k)}}{1-\nu_{12}^{(k)}\nu_{21}^{(k)}}, Q_{22}^{(k)} = \frac{E_2^{(k)}}{1-\nu_{12}^{(k)}\nu_{21}^{(k)}}, Q_{12}^{(k)} = \frac{\nu_{12}^{(k)}E_2^{(k)}}{1-\nu_{12}^{(k)}\nu_{21}^{(k)}}, Q_{66}^{(k)} = G_{12}^{(k)}, Q_{44}^{(k)} = G_{13}^{(k)}, Q_{55}^{(k)} = G_{23}^{(k)} \tag{22}$$

where $E_1, E_2, G_{13}, G_{23}, G_{12}, \nu_{12}$ are the engineering parameters of the *k*-th lamina. It should be noted that for a complete characterization of an orthotropic material, the parameters E_3, ν_{13}, ν_{23} have to be taken into account, as well-known. Following the virtual work principle in dynamic version, and remembering the Gauss-Codazzi relations for the shells of revolution (12), five equations of dynamic equilibrium in terms of internal actions can be written for the revolution shell element:

$$\begin{aligned}
&\frac{1}{R_\varphi}\frac{\partial N_\varphi}{\partial \varphi}+\frac{\partial N_{s\varphi}}{\partial s}+\left(N_\varphi-N_s\right)\frac{\cos\varphi}{R_0}+\frac{1}{R_\varphi}\left(\frac{1}{R_\varphi}\frac{\partial M_\varphi}{\partial \varphi}+\frac{\partial M_{s\varphi}}{\partial s}+\left(M_\varphi-M_s\right)\frac{\cos\varphi}{R_0}\right)+q_\varphi=\left(I_0+\frac{2I_1}{R_\varphi}+\frac{I_2}{R_\varphi^2}\right)\ddot{u}_\varphi\\
&+\left(I_1+\frac{I_2}{R_\varphi}\right)\ddot{\beta}_\varphi\,\frac{1}{R_\varphi}\frac{\partial N_{\varphi s}}{\partial \varphi}+\frac{\partial N_s}{\partial s}+\left(N_{\varphi s}+N_{s\varphi}\right)\frac{\cos\varphi}{R_0}+\frac{\sin\varphi}{R_0}\left(\frac{1}{R_\varphi}\frac{\partial M_{\varphi s}}{\partial \varphi}+\frac{\partial M_s}{\partial s}+\left(M_{\varphi s}+M_{s\varphi}\right)\frac{\cos\varphi}{R_0}\right)+q_s\\
&=\left(I_0+\frac{2I_1\sin\varphi}{R_0}+\frac{I_2\sin^2\varphi}{R_0^2}\right)\ddot{u}_s+\left(I_1+\frac{I_2\sin\varphi}{R_0}\right)\ddot{\beta}_s\,\frac{1}{R_\varphi}\frac{\partial T_\varphi}{\partial \varphi}+\frac{\partial T_s}{\partial s}+T_\varphi\frac{\cos\varphi}{R_0}-\frac{N_\varphi}{R_\varphi}-N_s\frac{\sin\varphi}{R_0}+q_n\\
&=I_0\ddot{w}\,\frac{1}{R_\varphi}\frac{\partial M_\varphi}{\partial \varphi}+\frac{\partial M_{s\varphi}}{\partial s}+\left(M_\varphi-M_s\right)\frac{\cos\varphi}{R_0}-T_\varphi+m_\varphi\\
&=\left(I_1+\frac{I_2}{R_\varphi}\right)\ddot{u}_\varphi+I_2\ddot{\beta}_\varphi\,\frac{1}{R_\varphi}\frac{\partial M_{\varphi s}}{\partial \varphi}+\frac{\partial M_s}{\partial s}+\left(M_{\varphi s}+M_{s\varphi}\right)\frac{\cos\varphi}{R_0}-T_s+m_s=\left(I_1+\frac{I_2\sin\varphi}{R_0}\right)\ddot{u}_s+I_2\ddot{\beta}_s
\end{aligned}\tag{23}$$

where:

$$\begin{aligned}
I_0&=\sum_{k=1}^{l}\int_{\zeta_k}^{\zeta_{k+1}}\rho^{(k)}\left(1+\frac{\zeta}{R_\varphi}\right)\left(1+\frac{\zeta\sin\varphi}{R_0}\right)\mathrm{d}\zeta+\frac{1}{3}\rho_F^+h_F^+\left(1+\frac{h}{2R_\varphi}\right)\left(1+\frac{h\sin\varphi}{2R_0}\right)+\frac{1}{3}\rho_F^-h_F^-\left(1-\frac{h}{2R_\varphi}\right)\left(1-\frac{h\sin\varphi}{2R_0}\right)\\
I_1&=\sum_{k=1}^{l}\int_{\zeta_k}^{\zeta_{k+1}}\rho^{(k)}\zeta\left(1+\frac{\zeta}{R_\varphi}\right)\left(1+\frac{\zeta\sin\varphi}{R_0}\right)\mathrm{d}\zeta+\frac{1}{3}\rho_F^+h_F^+\frac{h}{2}\left(1+\frac{h}{2R_\varphi}\right)\left(1+\frac{h\sin\varphi}{2R_0}\right)-\frac{1}{3}\rho_F^-h_F^-\frac{h}{2}\left(1-\frac{h}{2R_\varphi}\right)\left(1-\frac{h\sin\varphi}{2R_0}\right)\\
I_2&=\sum_{k=1}^{l}\int_{\zeta_k}^{\zeta_{k+1}}\rho^{(k)}\zeta^2\left(1+\frac{\zeta}{R_\varphi}\right)\left(1+\frac{\zeta\sin\varphi}{R_0}\right)\mathrm{d}\zeta+\frac{1}{3}\rho_F^+h_F^+\frac{h^2}{4}\left(1+\frac{h}{2R_\varphi}\right)\left(1+\frac{h\sin\varphi}{2R_0}\right)+\frac{1}{3}\rho_F^-h_F^-\frac{h^2}{4}\left(1-\frac{h}{2R_\varphi}\right)\left(1-\frac{h\sin\varphi}{2R_0}\right)
\end{aligned}\tag{24}$$

are the mass inertias and $\rho^{(k)}$ is the mass density of the material per unit of volume of the k-th ply, while ρ_F^+, h_F^+ and ρ_F^-, h_F^- are the mass density of the material per unit of volume and the thickness of the elastic foundation at the top and the bottom surface of the shell, respectively. The first three equations (23) represent translational equilibrium along meridian φ, circumferential s and normal ζ directions, while the last two are rotational equilibrium equations about the s and φ directions, respectively. Furthermore, the generalized external forces $q_\varphi, q_s, q_n, m_\varphi, m_s$ due to the external forces and the Winkler-Pasternak elastic foundation acting on the top and bottom surfaces of the shell can be evaluated using the static equivalence principle [12,79,83] and can be written on the reference surface of the doubly-curved shell as follows:

$$\begin{aligned}
q_\varphi&=q_\varphi^+\left(1+\frac{h}{2R_\varphi}\right)^2\left(1+\frac{h\sin\varphi}{2R_0}\right)+q_\varphi^-\left(1-\frac{h}{2R_\varphi}\right)^2\left(1-\frac{h\sin\varphi}{2R_0}\right)\\
&-\left(k_\varphi^+\left(1+\frac{h}{2R_\varphi}\right)^3\left(1+\frac{h\sin\varphi}{2R_0}\right)+k_\varphi^-\left(1-\frac{h}{2R_\varphi}\right)^3\left(1-\frac{h\sin\varphi}{2R_0}\right)\right)u_\varphi\\
&-\frac{h}{2}\left(k_\varphi^+\left(1+\frac{h}{2R_\varphi}\right)^2\left(1+\frac{h\sin\varphi}{2R_0}\right)-k_\varphi^-\left(1-\frac{h}{2R_\varphi}\right)^2\left(1-\frac{h\sin\varphi}{2R_0}\right)\right)\beta_\varphi\\
q_s&=q_s^+\left(1+\frac{h}{2R_\varphi}\right)^2\left(1+\frac{h\sin\varphi}{2R_0}\right)+q_s^-\left(1-\frac{h}{2R_\varphi}\right)^2\left(1-\frac{h\sin\varphi}{2R_0}\right)\\
&-\left(k_s^+\left(1+\frac{h}{2R_\varphi}\right)\left(1+\frac{h\sin\varphi}{2R_0}\right)^3+k_s^-\left(1-\frac{h}{2R_\varphi}\right)\left(1-\frac{h\sin\varphi}{2R_0}\right)^3\right)u_s\\
&-\frac{h}{2}\left(k_s^+\left(1+\frac{h}{2R_\varphi}\right)\left(1+\frac{h\sin\varphi}{2R_0}\right)^2-k_s^-\left(1-\frac{h}{2R_\varphi}\right)\left(1-\frac{h\sin\varphi}{2R_0}\right)^2\right)\beta_s
\end{aligned}$$

$$q_n = q_n^+\left(1+\frac{h}{2R_\varphi}\right)\left(1+\frac{h\sin\varphi}{2R_0}\right)+q_n^-\left(1-\frac{h}{2R_\varphi}\right)\left(1-\frac{h\sin\varphi}{2R_0}\right)-\left(k_n^+\left(1+\frac{h}{2R_\varphi}\right)\left(1+\frac{h\sin\varphi}{2R_0}\right)+k_n^-\left(1-\frac{h}{2R_\varphi}\right)\left(1-\frac{h\sin\varphi}{2R_0}\right)\right)w$$
$$+G_F^+\left(\frac{1}{R_\varphi^2\left(1+\frac{h}{2R_\varphi}\right)^2}\frac{\partial^2 w}{\partial\varphi^2}+\left(\frac{\cos\varphi}{R_\varphi\left(1+\frac{h}{2R_\varphi}\right)R_0\left(1+\frac{h\sin\varphi}{2R_0}\right)}-\frac{1}{R_\varphi^3\left(1+\frac{h}{2R_\varphi}\right)^3}\frac{\partial R_\varphi}{\partial\varphi}\right)\frac{\partial w}{\partial\varphi}\right.$$
$$\left.+\frac{1}{\left(1+\frac{h\sin\varphi}{2R_0}\right)^2}\frac{\partial^2 w}{\partial s^2}\right)\left(1+\frac{h}{2R_\varphi}\right)\left(1+\frac{h\sin\varphi}{2R_0}\right)$$
$$+G_F^-\left(\frac{1}{R_\varphi^2\left(1-\frac{h}{2R_\varphi}\right)^2}\frac{\partial^2 w}{\partial\varphi^2}+\left(\frac{\cos\varphi}{R_\varphi\left(1-\frac{h}{2R_\varphi}\right)R_0\left(1-\frac{h\sin\varphi}{2R_0}\right)}-\frac{1}{R_\varphi^3\left(1-\frac{h}{2R_\varphi}\right)^3}\frac{\partial R_\varphi}{\partial\varphi}\right)\frac{\partial w}{\partial\varphi}\right.$$
$$\left.+\frac{1}{\left(1-\frac{h\sin\varphi}{2R_0}\right)^2}\frac{\partial^2 w}{\partial s^2}\right)\left(1-\frac{h}{2R_\varphi}\right)\left(1-\frac{h\sin\varphi}{2R_0}\right)$$

$$m_\varphi = q_\varphi^+\frac{h}{2}\left(1+\frac{h}{2R_\varphi}\right)\left(1+\frac{h\sin\varphi}{2R_0}\right)-q_\varphi^-\frac{h}{2}\left(1-\frac{h}{2R_\varphi}\right)\left(1-\frac{h\sin\varphi}{2R_0}\right)$$
$$-\frac{h}{2}\left(k_\varphi^+\left(1+\frac{h}{2R_\varphi}\right)^2\left(1+\frac{h\sin\varphi}{2R_0}\right)-k_\varphi^-\left(1-\frac{h}{2R_\varphi}\right)^2\left(1-\frac{h\sin\varphi}{2R_0}\right)\right)u_\varphi$$
$$-\frac{h^2}{4}\left(k_\varphi^+\left(1+\frac{h}{2R_\varphi}\right)\left(1+\frac{h\sin\varphi}{2R_0}\right)+k_\varphi^-\left(1-\frac{h}{2R_\varphi}\right)\left(1-\frac{h\sin\varphi}{2R_0}\right)\right)\beta_\varphi$$

$$m_s = q_s^+\frac{h}{2}\left(1+\frac{h}{2R_\varphi}\right)\left(1+\frac{h\sin\varphi}{2R_0}\right)-q_s^-\frac{h}{2}\left(1-\frac{h}{2R_\varphi}\right)\left(1-\frac{h\sin\varphi}{2R_0}\right)$$
$$-\frac{h}{2}\left(k_s^+\left(1+\frac{h}{2R_\varphi}\right)\left(1+\frac{h\sin\varphi}{2R_0}\right)^2-k_s^-\left(1-\frac{h}{2R_\varphi}\right)\left(1-\frac{h\sin\varphi}{2R_0}\right)^2\right)u_s \qquad (25)$$
$$-\frac{h^2}{4}\left(k_s^+\left(1+\frac{h}{2R_\varphi}\right)\left(1+\frac{h\sin\varphi}{2R_0}\right)+k_s^-\left(1-\frac{h}{2R_\varphi}\right)\left(1-\frac{h\sin\varphi}{2R_0}\right)\right)\beta_s$$

where $q_\varphi^+, q_\varphi^-, q_s^+, q_s^-, q_n^+, q_n^-$ are the external forces and $k_\varphi^+, k_\varphi^-, k_s^+, k_s^-, k_n^+, k_n^-$ are the Winkler elastic stiffnesses in the three principal directions φ, s, ζ at the top and the bottom surface of the shell, respectively; G_F^+, G_F^- are the shear modulus of the Pasternak elastic foundation at the top and the bottom surface of the shell. It is worth noting that the formulation of the Winkler-Pasternak foundation is based on the first-order approximated assumption that the foundation is a homogeneous material of uniform thickness h_F^+, h_F^- [12,79,83]. Furthermore, as additional hypothesis, the effect of the damping of the foundation is neglected. The three basic set of equations, namely the kinematic (18), constitutive (19) and motion (23) equations may be combined to give the fundamental system of equations, also known as the governing system of equations. By replacing the kinematic equations (18) into the constitutive equations (19) and the result of this substitution into the motion equations (23), the complete equations of motion in terms of displacements can be written as:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{bmatrix} \begin{bmatrix} u_\varphi \\ u_s \\ w \\ \beta_\varphi \\ \beta_s \end{bmatrix} + \begin{bmatrix} q_\varphi \\ q_s \\ q_n \\ m_\varphi \\ m_s \end{bmatrix} = \begin{bmatrix} I_{0\varphi} & 0 & 0 & I_{1\varphi} & 0 \\ 0 & I_{0s} & 0 & 0 & I_{1s} \\ 0 & 0 & I_0 & 0 & 0 \\ I_{1\varphi} & 0 & 0 & I_2 & 0 \\ 0 & I_{1s} & 0 & 0 & I_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_\varphi \\ \ddot{u}_s \\ \ddot{w} \\ \ddot{\beta}_\varphi \\ \ddot{\beta}_s \end{bmatrix} \tag{26}$$

where $L_{ij}, i, j = 1,\cdots,5$ are the equilibrium operators and the new mass inertias are defined as follows:

$$\begin{aligned} I_{0\varphi} &= I_0 + \frac{2I_1}{R_\varphi} + \frac{I_2}{R_\varphi^2}, I_{1\varphi} = I_1 + \frac{I_2}{R_\varphi} \\ I_{0s} &= I_0 + \frac{2I_1 \sin\varphi}{R_0} + \frac{I_2 \sin^2\varphi}{R_0^2}, I_{1s} = I_1 + \frac{I_2 \sin\varphi}{R_0} \end{aligned} \tag{27}$$

It is worth noting that, differently from previous works by Tornabene [77,78], the mass matrix and the equilibrium operators L_{ij}, introduced in Equation (27), have changed due to the choice of using the new kinematical model (18). Furthermore, the second derivative of the principal radius $R_\varphi(\varphi)$ respect to φ has to be evaluated, as it can be inferred from the explicit form of the equilibrium operators L_{ij}. Three types of boundary conditions are considered, namely the fully clamped edge boundary condition (*C*), soft simply supported edge boundary conditions (*S*) and the free edge boundary condition (*F*). The equations describing the boundary conditions can be written as follows:

Clamped edge boundary conditions (*C*)

$$\begin{aligned} &u_\varphi = u_s = w = \beta_\varphi = \beta_s = 0 \\ &\text{at } \varphi = \varphi_0 \text{ or } \varphi = \varphi_1, 0 \le s \le s_0 \end{aligned} \tag{28}$$

$$\begin{aligned} &u_\varphi = u_s = w = \beta_\varphi = \beta_s = 0 \\ &\text{at } s = 0 \text{ or } s = s_0, \varphi_0 \le \varphi \le \varphi_1 \end{aligned} \tag{29}$$

Soft simply supported edge boundary conditions (*S*)

$$\begin{aligned} &u_s = w = \beta_s = 0, N_\varphi + \frac{M_\varphi}{R_\varphi} = M_\varphi = 0 \\ &\text{at } \varphi = \varphi_0 \text{ or } \varphi = \varphi_1, 0 \le s \le s_0 \end{aligned} \tag{30}$$

$$\begin{aligned} &u_\varphi = w = \beta_\varphi = 0, N_s + \frac{M_s}{R_s} = M_s = 0 \\ &\text{at } s = 0 \text{ or } s = s_0, \varphi_0 \le \varphi \le \varphi_1 \end{aligned} \tag{31}$$

Free edge boundary conditions (*F*)

$$\begin{aligned} &N_\varphi + \frac{M_\varphi}{R_\varphi} = N_{\varphi s} + M_{\varphi s}\frac{\sin\varphi}{R_0} = T_\varphi = M_\varphi = M_{\varphi s} = 0 \\ &\text{at } \varphi = \varphi_0 \text{ or } \varphi = \varphi_1, 0 \le s \le s_0 \end{aligned} \tag{32}$$

$$\begin{aligned} &N_s + M_s\frac{\sin\varphi}{R_0} = N_{s\varphi} + \frac{M_{s\varphi}}{R_\varphi} = T_s = M_s = M_{s\varphi} = 0 \\ &\text{at } s = 0 \text{ or } s = s_0, \varphi_0 \le \varphi \le \varphi_1 \end{aligned} \tag{33}$$

where $\varphi_0 = \hat{\varphi}_1$ and $\varphi_1 = \hat{\varphi}_N$. In addition to the external boundary conditions, the kinematic and physical compatibility conditions should be satisfied at the common closing meridians with $s = 0, 2\pi R_0$, if a complete shell of revolution is considered. The kinematic compatibility conditions include the continuity of displacements. The physical compatibility conditions can be only represented by the five continuous conditions for the generalized stress resultants. To consider complete revolute shells, it is necessary to implement the kinematic and physical compatibility conditions between the two computational meridians with $s = 0$ and with $s_0 = 2\pi R_0$:

Kinematic compatibility conditions along the closing meridian $(s = 0, 2\pi R_0)$

$$\begin{aligned} &u_\varphi(\varphi,0,t) = u_\varphi(\varphi,s_0,t), u_s(\varphi,0,t) = u_s(\varphi,s_0,t), \\ &w(\varphi,0,t) = w(\varphi,s_0,t), \\ &\beta_\varphi(\varphi,0,t) = \beta_\varphi(\varphi,s_0,t), \beta_s(\varphi,0,t) \\ &= \beta_s(\varphi,s_0,t) \text{ for } \varphi_0 \le \varphi \le \varphi_1 \end{aligned} \tag{34}$$

Physical compatibility conditions along the closing meridian $(s = 0, 2\pi R_0)$

$$\begin{aligned} &N_s(\varphi,0,t) + M_s(\varphi,0,t)\frac{\sin\varphi}{R_0} \\ &= N_s(\varphi,s_0,t) + M_s(\varphi,s_0,t)\frac{\sin\varphi}{R_0}, \\ &N_{s\varphi}(\varphi,0,t) + \frac{M_{s\varphi}(\varphi,0,t)}{R_\varphi} \\ &= N_{s\varphi}(\varphi,s_0,t) + \frac{M_{s\varphi}(\varphi,s_0,t)}{R_\varphi}, \\ &T_s(\varphi,0,t) = T_s(\varphi,s_0,t), M_s(\varphi,0,t) = M_s(\varphi,s_0,t), \\ &M_{s\varphi}(\varphi,0,t) = M_{s\varphi}(\varphi,s_0,t) \text{ for } \varphi_0 \le \varphi \le \varphi_1 \end{aligned} \tag{35}$$

where $\varphi_0 = \hat{\varphi}_1$ and $\varphi_1 = \hat{\varphi}_N$. In an analogous way, in order to consider a toroidal shell of revolution, it is necessary to implement the kinematic and physical compatibility conditions between the two computational parallels with $\varphi_0 = 0$ and with $\varphi_1 = 2\pi$:

Kinematic compatibility conditions along the closing parallel $(\varphi = 0, 2\pi)$

$$\begin{aligned}
&u_\varphi(0,s,t) = u_\varphi(2\pi,s,t), u_s(0,s,t) = u_s(2\pi,s,t),\\
&w(0,s,t) = w(2\pi,s,t), \beta_\varphi(0,s,t) = \beta_\varphi(2\pi,s,t),\\
&\beta_s(0,s,t) = \beta_s(2\pi,s,t) \text{ for } 0 \le s \le s_0
\end{aligned} \tag{36}$$

Physical compatibility conditions along the closing parallel $(\varphi = 0, 2\pi)$

$$\begin{aligned}
&N_\varphi(0,s,t) + \frac{M_\varphi(0,s,t)}{R_\varphi}\\
&= N_\varphi(2\pi,s,t) + \frac{M_\varphi(2\pi,s,t)}{R_\varphi},\\
&N_{\varphi s}(0,s,t) + M_{\varphi s}(0,s,t)\frac{\sin\varphi}{R_0}\\
&= N_{\varphi s}(2\pi,s,t) + M_{\varphi s}(2\pi,s,t)\frac{\sin\varphi}{R_0},\\
&T_\varphi(0,s,t) = T_\varphi(2\pi,s,t), M_\varphi(0,s,t) = M_\varphi(2\pi,s,t),\\
&M_{\varphi s}(0,s,t) = M_{\varphi s}(2\pi,s,t) \text{ for } 0 \le s \le s_0
\end{aligned} \tag{37}$$

3. Numerical Implementation

The Generalized Differential Quadrature method will be used to discretize the derivatives in the governing equations in terms of displacements as well as boundary and compatibility conditions. Since a review of the GDQ Method is presented in Tornabene [65,83], the same approach is used in the present work about the GDQ technique. The Chebyshev-Gauss-Lobatto (C-G-L) grid distribution is assumed. Since the co-ordinates of the grid points of the reference surface in the φ direction are introduced in Equation (9), then the co-ordinates of the grid points in the s direction are the following:

$$\begin{aligned}
&s_j = \left(1 - \cos\left(\frac{j-1}{M-1}\pi\right)\right)\frac{s_0}{2}\\
&\text{for } j = 1,2,\cdots,M, \text{for } s \in [0, s_0] (\text{with } s \le \vartheta R_0)
\end{aligned} \tag{38}$$

where M is the total number of sampling points used to discretize the domain in s direction of the doubly-curved shell. It has been proven that for the Lagrange interpolating polynomials, the Chebyshev-Gauss-Lobatto sampling points rule guarantees convergence and efficiency to the GDQ technique [57-59,63,83]. For the static analysis, when the inertias are set to zero, the GDQ procedure enables to write the equations of equilibrium (26) in discrete form, transforming each space derivative into a weighted sum of node values of independent variables. Each approximated equation is valid in a single sampling point. Thus, the whole system of differential equations has been discretized and the global assembling leads to the following set of linear algebraic equations:

$$\left[\begin{array}{c|c} \boldsymbol{K}_{bb} & \boldsymbol{K}_{bd} \\ \hline \boldsymbol{K}_{db} & \boldsymbol{K}_{dd} \end{array}\right]\left[\begin{array}{c} \boldsymbol{\delta}_b \\ \hline \boldsymbol{\delta}_d \end{array}\right] = \left[\begin{array}{c} \boldsymbol{f}_b \\ \hline \boldsymbol{f}_d \end{array}\right] \tag{39}$$

In the above mentioned matrices and vectors, the partitioning is set forth by subscripts b and d, referring to the system degrees of freedom and standing for boundary and domain, respectively. In this sense, b-equations represent the discrete boundary conditions, which are valid only for the points lying on constrained edges of the shell; while d-equations are the equilibrium equations, assigned to the interior nodes. In order to make the computation more efficient, static condensation of non-domain degrees of freedom is performed:

$$\left(\boldsymbol{K}_{dd} - \boldsymbol{K}_{db}\boldsymbol{K}_{bb}^{-1}\boldsymbol{K}_{bd}\right)\boldsymbol{\delta}_d = \boldsymbol{f}_d - \boldsymbol{K}_{db}\boldsymbol{K}_{bb}^{-1}\boldsymbol{f}_b \tag{40}$$

The deflection of the structures considered can be determined by solving the linear algebraic problem (40). In particular, the solution procedure by means of the GDQ technique has been implemented in a MATLAB code. Otherwise, when the external forces $q_\varphi^+, q_\varphi^-, q_s^+, q_s^-, q_n^+, q_n^-$ are set to zero, the free vibration of laminated composite doubly-curved shells and panels of revolution can be studied. Using the method of variable separation, it is possible to seek solutions that are harmonic in time and whose frequency is $f = \omega/2\pi$. The displacement field can be written as follows:

$$\begin{aligned}
&u_\varphi(\varphi,s,t) = U^\varphi(\varphi,s)\mathrm{e}^{i\omega t}\\
&u_s(\varphi,s,t) = U^s(\varphi,s)\mathrm{e}^{i\omega t}\\
&w(\varphi,s,t) = W(\varphi,s)\mathrm{e}^{i\omega t}\\
&\beta_\varphi(\varphi,s,t) = B^\varphi(\varphi,s)\mathrm{e}^{i\omega t}\\
&\beta_s(\varphi,s,t) = B^s(\varphi,s)\mathrm{e}^{i\omega t}
\end{aligned} \tag{41}$$

where the vibration spatial amplitude values U^φ, U^s, W, B^φ, B^s fulfill the fundamental differential system. Thus, the whole system of differential equations has been discretized and the global assembling leads to the following linear eigenvalue problem:

$$\left[\begin{array}{c|c} \boldsymbol{K}_{bb} & \boldsymbol{K}_{bd} \\ \hline \boldsymbol{K}_{db} & \boldsymbol{K}_{dd} \end{array}\right]\left[\begin{array}{c} \boldsymbol{\delta}_b \\ \hline \boldsymbol{\delta}_d \end{array}\right] = \omega^2\left[\begin{array}{c|c} \boldsymbol{0} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{M}_{dd} \end{array}\right]\left[\begin{array}{c} \boldsymbol{\delta}_b \\ \hline \boldsymbol{\delta}_d \end{array}\right] \tag{42}$$

In order to make the computation more efficient, kinematic condensation of non-domain degrees of freedom is performed:

$$\left(\boldsymbol{K}_{dd} - \boldsymbol{K}_{db}\left(\boldsymbol{K}_{bb}\right)^{-1}\boldsymbol{K}_{bd}\right)\boldsymbol{\delta}_d = \omega^2\boldsymbol{M}_{dd}\boldsymbol{\delta} \tag{43}$$

The natural frequencies of the structure considered can be determined by solving the standard eigenvalue problem (43). In particular, the solution procedure by means of the GDQ technique has been implemented in a MATLAB code. Finally, the results in terms of frequencies are obtained using the *eigs* function of MATLAB software. More details regarding the way to obtained the Equations (40) and (43) can be found in the previous works [57-59,63,65-68,83]. It is worth noting that, with the present approach, differing from the finite element method, no integration occurs prior to the global assembly of the linear system, and this approach leads to a further computational cost saving in favor of the Differential Quadrature technique.

4. Numerical Results

In the present paragraph, some results and considerations about the static analysis and the free vibration problem of laminated composite doubly-curved shells and panels of revolution are presented. The analysis has been carried out by means of numerical procedures illustrated above. One of the aims of this paper is to compare results obtained through the GDQ method with the ones obtained through finite element techniques. In order to verify the accuracy of the present method, some comparisons and tests have been performed. Extensive attempts to validate the numerical procedure have been made for the isotropic and anisotropic cases and can be found in the Ph.D. Thesis by Tornabene [57] and in the book by Tornabene [83]. In this work, the static deflection and the frequency parameters evaluated by the present formulation are in good agreement with the results obtained with the finite element method. The geometrical boundary conditions for a panel are identified by the following convention. For example, symbolism CSCF shows that the edges $\varphi = \varphi_{0d}, s = 0, \varphi = \varphi_1, s = s_0$ are clamped, simply supported, clamped and free, respectively. On the contrary, for a complete shell of revolution or for a toroidal shell, symbolism CF shows that the edges $\varphi = \varphi_0$ and $\varphi = \varphi_1$ or $s = 0$ and $s = s_0$ are clamped and free, respectively. The missing boundary conditions are the kinematic and physical compatibility conditions that are applied at the same closing meridians for $s = 0$ and $s_0 = 2\pi R_0$ or at the same closing parallels for $\varphi = 0$ and $\varphi = 2\pi$, respectively. **Table 1** presents the static deflection at the point $A = (90^\circ, 0^\circ)$ for a SSSS spherical panel resting on elastic foundation and subjected to a uniformly distributed load $q_n^+ = -10000$ Pa at the top surface by considering different lamination schemes. As can be seen, the numerical results show an excellent agreement for all the cases considered. GDQ results are compared with the FEM results obtained with Straus 7 commercial software using 8 node parabolic shell elements. In order to illustrate the effect of the foundation **Figures 4** and **5** show the stress resultants for a $(30/45/70)$ CCCC spherical panel subjected to a uniformly distributed load $q_n^+ = -10000$ Pa at the top surface. **Figure 4** illustrates the stress resultants obtained without considering the foundation, while **Figure 5** presents the same quantities obtained considering the elastic foundation. The geometrical and material properties are the same reported in

Table 1. Static deflection for a SSSS spherical panel at the point $A = (90^\circ, 0^\circ)$ resting on elastic foundation and subjected to a uniformly distributed load $q_n^+ = -10000$ Pa at the top surface.

Foundation properties: $\rho_F^- = 0\ \mathrm{kg/m^3}$, $h_F^- = 0$ m, $G_F^- = 0\ \mathrm{N/m}$, $k_\varphi^- = k_s^- = 0\ \mathrm{N/m^3}$						
Lamination scheme	$k_n^- = 0\ \mathrm{N/m^3}$		$k_n^- = 1.5\times10^7\ \mathrm{N/m^3}$		$k_n^- = 7.5\times10^7\ \mathrm{N/m^3}$	
	GDQ-GST 31×31	Straus 100×100 (8 nodes)	GDQ-GST 31×31	Straus 100×100 (8 nodes)	GDQ-GST 31×31	Straus 100×100 (8 nodes)
Isotropic	−1.570E−04	−1.571E−04	−1.275E−04	−1.274E−04	−7.281E−05	−7.246E−05
(0/90)	−5.911E−04	−5.862E−04	−3.257E−04	−3.229E−04	−1.129E−04	−1.119E−04
(0/90/0)	−5.922E−04	−5.875E−04	−3.263E−04	−3.235E−04	−1.131E−04	−1.121E−04
$(0/90)_{as}$	−5.911E−04	−5.863E−04	−3.263E−04	−3.235E−04	−1.130E−04	−1.120E−04
$(0/90)_s$	−5.909E−04	−5.862E−04	−3.264E−04	−3.236E−04	−1.130E−04	−1.120E−04
(30)	−4.589E−04	−4.418E−04	−2.503E−04	−2.444E−04	−9.899E−05	−9.758E−05
(30/45)	−4.476E−04	−4.276E−04	−2.303E−04	−2.237E−04	−9.102E−05	−8.940E−05
(30/45/70)	−4.413E−04	−4.253E−04	−2.420E−04	−2.361E−04	−9.507E−05	−9.351E−05
(30/45/70/90)	−3.859E−04	−3.764E−04	−2.364E−04	−2.319E−04	−9.686E−05	−9.553E−05

*Geometric characteristics: $R_\varphi = R_s = 10$ m, $R_b = 0$ m, $\varphi \in [60^\circ, 120^\circ]$, $\vartheta \in [-30^\circ, 30^\circ]$, $h = 0.09$ m, Isotropic material: $E = 73$ GPa, $\nu = 0.3$, $\rho = 2700\ \mathrm{kg/m^3}$, Orthotropic material: $E_1 = 137.9$ GPa, $E_2 = 8.96$ GPa, $G_{12} = G_{13} = 7.1$ GPa, $G_{23} = 6.21$ GPa, $\nu_{12} = 0.3$, $\rho = 1450\ \mathrm{kg/m^3}$.

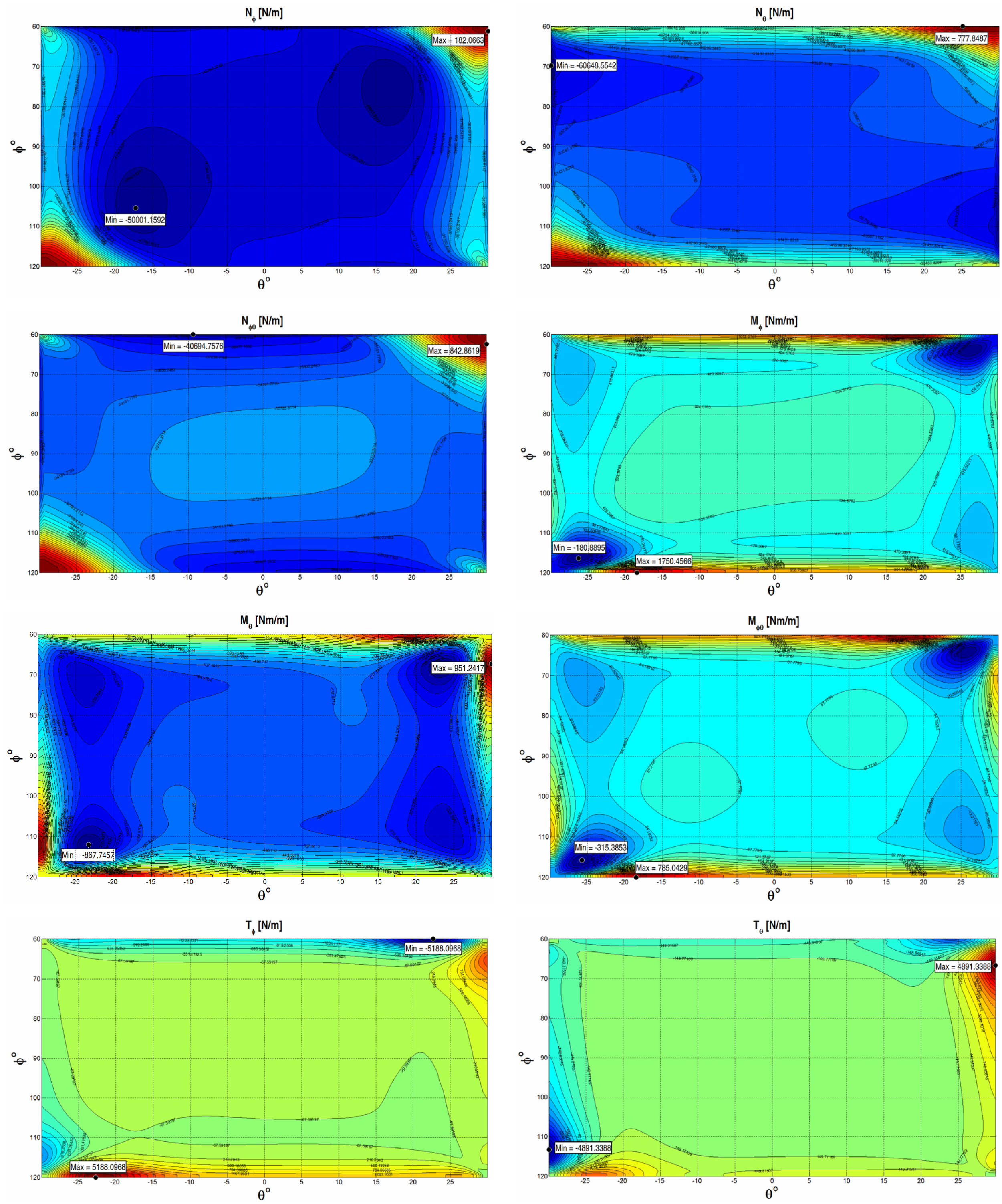

Figure 4. Stress resultants for a $(30/45/70)$ CCCC spherical panel with a uniformly distributed load $q_n^+ = -10000$ Pa at the top surface and without elastic foundation.

Table 1. As can be seen from the figures, the effect of the foundation reduces the stress resultants as expected. In **Tables 2-4**, the results, in terms of first ten frequencies obtained by the GDQ Method for the General Shell Theory (GST) presented above with and without elastic foundations, are compared with the FEM results obtained with Straus 7 commercial software. The details regarding the geometry and the material properties are reported in

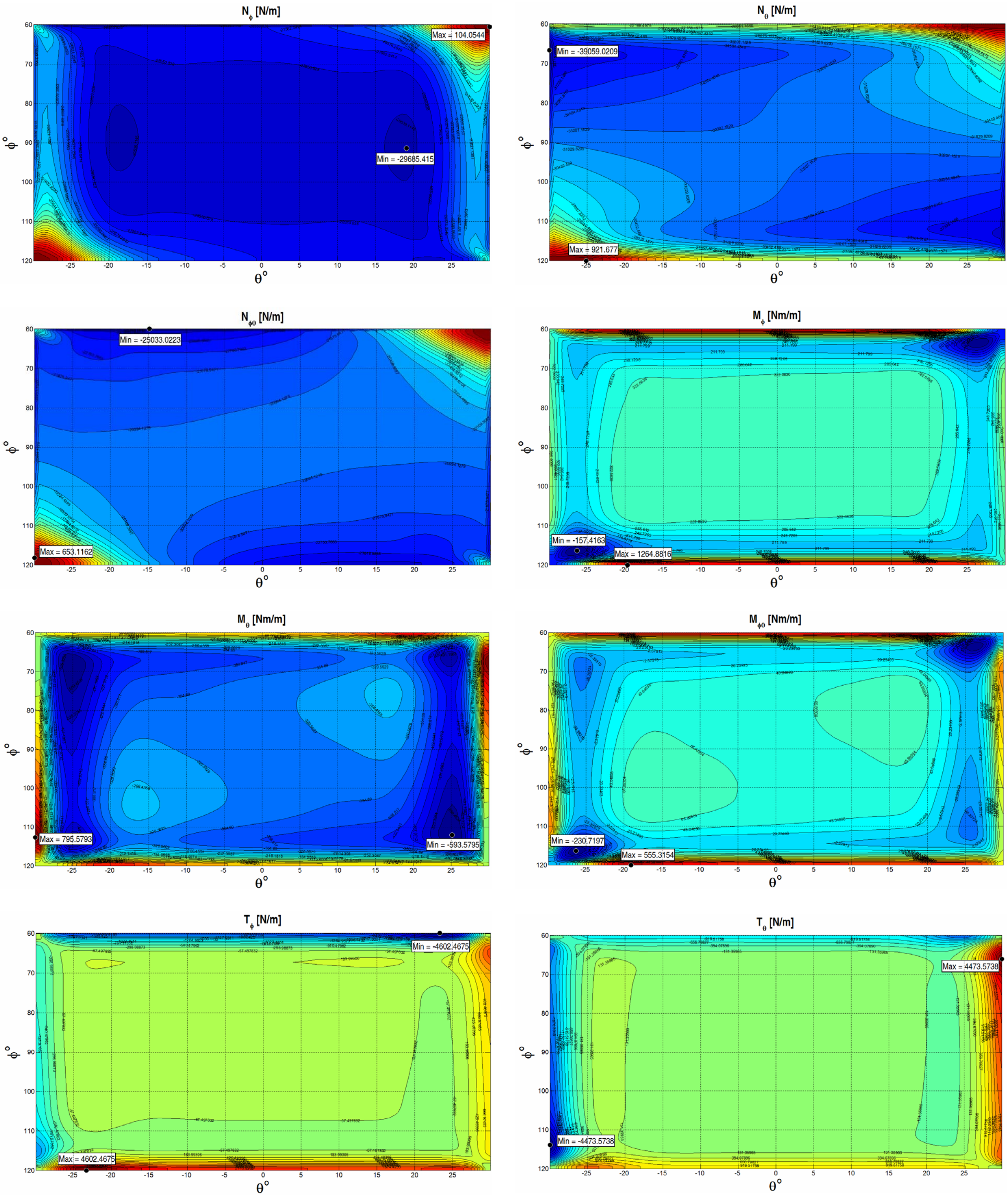

Figure 5. Stress resultants for a $(30/45/70)$ CCCC spherical panel with a uniformly distributed load $q_n^+ = -10000$ Pa at the top surface resting on Winkler-Pasternak elastic foundation: $k_n^- = 7.5\times10^7\ \mathrm{N/m^3}$ and $\rho_F^- = h_F^- = G_F^- = k_\varphi^- = k_s^- = 0$.

tables. For the present GDQ results, the grid distributions (9) and (38) with $N = M = 31$ have been considered. **Tables 2-4** present the first ten frequencies for a SSSS spherical panel characterized by $(0/90)$, $(0/90/0)$ and $(30/45/70)$ lamination scheme, respectively. As can be seen, the numerical results from the GDQ methodology are very close to those obtained by the commercial program and show an excellent agreement. **Tables 5-7** pre-

Table 2. First ten frequencies for a $(0/90)$ SSSS spherical panel resting on elastic foundation.

Mode [Hz]	$k_n^- = 0\,\mathrm{N/m^3}$ GDQ-GST 31×31	$k_n^- = 0\,\mathrm{N/m^3}$ Straus 100×100 (8 nodes)	$k_n^- = 1.5\times 10^7\,\mathrm{N/m^3}$ GDQ-GST 31×31	$k_n^- = 1.5\times 10^7\,\mathrm{N/m^3}$ Straus 100×100 (8 nodes)	$k_n^- = 7.5\times 10^7\,\mathrm{N/m^3}$ GDQ-GST 31×31	$k_n^- = 7.5\times 10^7\,\mathrm{N/m^3}$ Straus 100×100 (8 nodes)
Foundation properties: $\rho_F^- = 0\,\mathrm{kg/m^3}$, $h_F^- = 0$ m, $G_F^- = 0\,\mathrm{N/m}$, $k_\varphi^- = k_s^- = 0\,\mathrm{N/m^3}$						
f_1	59.775	60.016	77.703	78.035	96.630	97.056
f_2	64.413	64.661	83.209	83.552	111.257	111.754
f_3	69.263	69.518	87.167	87.516	125.527	126.086
f_4	69.527	69.790	87.285	87.604	134.209	134.795
f_5	70.561	70.777	87.554	87.908	135.981	136.554
f_6	70.612	70.866	88.298	88.644	136.490	137.085
f_7	73.148	73.425	89.717	90.078	137.712	138.301
f_8	75.950	76.275	92.796	93.200	137.805	138.399
f_9	76.739	76.996	93.436	93.784	137.961	138.549
f_{10}	79.349	79.641	95.537	95.914	141.355	141.981

*Geometric characteristics: $R_\varphi = R_s = 10$ m, $R_b = 0$ m, $\varphi \in [60^\circ, 120^\circ]$, $\vartheta \in [-30^\circ, 30^\circ]$, $h = 0.09$ m, Orthotropic material: $E_1 = 137.9$ GPa, $E_2 = 8.96$ GPa, $G_{12} = G_{13} = 7.1$ GPa, $G_{23} = 6.21$ GPa, $\nu_{12} = 0.3$, $\rho = 1450\,\mathrm{kg/m^3}$.

Table 3. First ten frequencies for a $(0/90/0)$ SSSS spherical panel resting on elastic foundation.

Mode [Hz]	$k_n^- = 0\,\mathrm{N/m^3}$ GDQ-GST 31×31	$k_n^- = 0\,\mathrm{N/m^3}$ Straus 100×100 (8 nodes)	$k_n^- = 1.5\times 10^7\,\mathrm{N/m^3}$ GDQ-GST 31×31	$k_n^- = 1.5\times 10^7\,\mathrm{N/m^3}$ Straus 100×100 (8 nodes)	$k_n^- = 7.5\times 10^7\,\mathrm{N/m^3}$ GDQ-GST 31×31	$k_n^- = 7.5\times 10^7\,\mathrm{N/m^3}$ Straus 100×100 (8 nodes)
Foundation properties: $\rho_F^- = 0\,\mathrm{kg/m^3}$, $h_F^- = 0$ m, $G_F^- = 0\,\mathrm{N/m}$, $k_\varphi^- = k_s^- = 0\,\mathrm{N/m^3}$						
f_1	59.914	60.144	77.848	78.173	96.630	97.056
f_2	65.246	65.463	83.870	84.190	111.179	111.678
f_3	68.782	68.984	86.022	86.330	125.657	126.215
f_4	72.096	72.316	89.482	89.802	134.655	135.228
f_5	72.494	72.754	89.668	90.019	135.572	136.139
f_6	73.792	74.062	90.991	91.351	137.496	138.082
f_7	75.771	76.021	91.902	92.244	138.703	139.276
f_8	79.808	80.087	95.985	96.351	139.605	140.196
f_9	81.940	82.157	96.630	97.055	140.049	140.647
f_{10}	83.242	83.452	97.682	97.995	143.482	144.083

*Geometric characteristics: $R_\varphi = R_s = 10$ m, $R_b = 0$ m, $\varphi \in [60^\circ, 120^\circ]$, $\vartheta \in [-30^\circ, 30^\circ]$, $h = 0.09$ m, Orthotropic material: $E_1 = 137.9$ GPa, $E_2 = 8.96$ GPa, $G_{12} = G_{13} = 7.1$ GPa, $G_{23} = 6.21$ GPa, $\nu_{12} = 0.3$, $\rho = 1450\,\mathrm{kg/m^3}$.

Table 4. First ten frequencies for a $(30/45/70)$ SSSS spherical panel resting on elastic foundation.

Foundation properties: $\rho_F^- = 0\,\mathrm{kg/m^3}$, $h_F^- = 0$ m, $G_F^- = 0\,\mathrm{N/m}$, $k_\varphi^- = k_s^- = 0\,\mathrm{N/m^3}$						
Mode [Hz]	$k_n^- = 0\,\mathrm{N/m^3}$		$k_n^- = 1.5\times10^7\,\mathrm{N/m^3}$		$k_n^- = 7.5\times10^7\,\mathrm{N/m^3}$	
	GDQ-GST 31×31	Straus 100×100 (8 nodes)	GDQ-GST 31×31	Straus 100×100 (8 nodes)	GDQ-GST 31×31	Straus 100×100 (8 nodes)
f_1	49.808	50.761	72.378	73.212	113.366	114.996
f_2	50.298	51.252	72.504	73.353	122.924	124.126
f_3	58.625	59.540	78.395	79.266	126.787	127.684
f_4	59.718	60.646	79.653	80.515	130.940	131.884
f_5	63.879	64.983	82.107	83.137	132.074	132.994
f_6	65.655	66.760	84.378	85.393	134.626	135.592
f_7	68.939	69.923	85.701	86.673	135.405	136.422
f_8	70.901	71.688	88.598	89.372	138.339	139.210
f_9	75.107	76.139	91.808	92.794	139.764	140.811
f_{10}	77.134	78.199	92.876	93.925	139.863	140.883

*Geometric characteristics: $R_\varphi = R_s = 10$ m, $R_b = 0$ m, $\varphi \in [60^\circ, 120^\circ]$, $\vartheta \in [-30^\circ, 30^\circ]$, $h = 0.09$ m, Orthotropic material: $E_1 = 137.9$ GPa, $E_2 = 8.96$ GPa, $G_{12} = G_{13} = 7.1$ GPa, $G_{23} = 6.21$ GPa, $\nu_{12} = 0.3$, $\rho = 1450\,\mathrm{kg/m^3}$.

sent new results regarding different anisotropic doubly-curved shells and panels of revolution resting on Winkler-Pasternak elastic foundations. **Tables 5-7** are divided in different columns where each effect of the elastic foundation is considered. At first, the effect of the Winkler modulus on free vibrations has been investigated and compared to the case without the elastic foundation. As can be seen from results, the Winkler foundation increases the stiffness of the structure. The same effect is shown by introducing the Pasternak shear modulus. In addition to these two effects the inertia of the foundation is considered. As expected, the inertia effect increases the mass of the structure and reduces the frequencies. Finally, the Winkler foundation in meridian and circumferential directions is introduced. As can be inferred, also in this case the stiffness of the structure is increased. **Table 5** presents the effects under consideration for the CC complete shell characterized by the $(30/45)$ lamination scheme. In this case, the foundation acts on the top surface of the shell. Differently from previous case, **Tables 6** and **7** show the first ten frequencies for the $(0/30)$ CCCC laminated composite panel and for the $(30/60)$ FC laminated composite shell resting on an elastic foundation acting on the bottom surface. Also for these cases, the effect of the initial curvature is considered by comparing the present General Shell Theory (GST) and the Reissner-Mindlin (RM) theory. Thus, as regarding the influence of the initial curvature, it is worth noting that the difference between the two theories considered is low for all the laminated composite doubly-curved structures analyzed. Finally, in **Figures 6** and **7**, there are reported the first six mode shapes for some of the structures considered above. In particular, for the complete shells of revolution there are some symmetrical mode shapes due to the symmetry of the problem considered in 3D space. In these cases, the symmetrical mode shapes are summarized in one figure. The mode shapes of all the structures have been evaluated by the authors. By using the authors' MATLAB code, these mode shapes have been reconstructed in three-dimensional view by means of considering the displacement field (17) after solving the eigenvalue problem (43). In addition, Hosseini-Hashemi *et al*. [86] (see **table 3**) and Zhao and Liew [87] (see **Table 6**) have compared their results, obtained with a semi analytical method and produced by a meshless method, with those results presented in the articles by Tornabene [65] and by Tornabene *et al*. [69], respectively. Since the code used to obtain the previous results by Tornabene [65] and Tornabene *et al*. [69] is similar to the code used to obtain all the results presented in this paper, the results of the work by Hosseini-Hashemi *et al*. [86] and by Zhao and Liew [87] represent other proofs of the validity and the accuracy of the present procedure. As shown, the exact results by Hosseini-Hashemi *et al*. [86] and the numerical results by Zhao and Liew [84] are in good agreement with those reported by Tornabene [65] and by

Table 5. First ten frequencies for a $(30/45)$ CC shell resting on Winkler-Pasternak elastic foundation.

Foundation properties: $\rho_F^+ = 1800\,\mathrm{kg/m^3}$, $h_F^+ = 0.1$ m, $k_n^+ = 1.5\times10^7\ \mathrm{N/m^3}$, $G_F^+ = 5\times10^5\ \mathrm{N/m}$, $k_\varphi^+ = k_s^+ = 0.75\times10^7\ \mathrm{N/m^3}$										
Mode [Hz]	$k_n^+ = 0, k_\varphi^+ = 0,$ $k_s^+ = 0, G_F^+ = 0,$ $\rho_F^+ = 0, h_F^+ = 0$		$k_n^+ \neq 0, k_\varphi^+ = 0,$ $k_s^+ = 0, G_F^+ = 0,$ $\rho_F^+ = 0, h_F^+ = 0$		$k_n^+ \neq 0, k_\varphi^+ = 0,$ $k_s^+ = 0, G_F^+ \neq 0,$ $\rho_F^+ = 0, h_F^+ = 0$		$k_n^+ \neq 0, k_\varphi^+ = 0,$ $k_s^+ = 0, G_F^+ \neq 0,$ $\rho_F^+ \neq 0, h_F^+ \neq 0$		$k_n^+ \neq 0, k_\varphi^+ \neq 0,$ $k_s^+ \neq 0, G_F^+ \neq 0,$ $\rho_F^+ \neq 0, h_F^+ \neq 0$	
	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM
f_1	439.664	439.768	440.999	441.043	441.317	441.345	364.415	364.341	365.394	365.339
f_2	447.296	447.842	449.607	450.144	450.176	450.711	375.370	375.795	375.720	376.149
f_3	447.296	447.842	449.607	450.144	450.176	450.711	375.370	375.795	375.720	376.149
f_4	461.762	461.978	463.248	463.374	463.637	463.779	386.898	387.106	387.431	387.621
f_5	461.762	462.406	464.276	464.916	464.911	465.549	389.309	389.840	389.477	390.008
f_6	461.907	462.406	464.276	464.916	464.911	465.549	389.309	389.840	389.477	390.008
f_7	487.146	487.753	489.657	490.261	490.422	491.024	411.574	412.079	411.661	412.166
f_8	487.146	487.753	489.657	490.261	490.422	491.024	411.574	412.079	411.661	412.166
f_9	523.557	524.045	525.718	525.638	525.902	525.823	436.735	436.676	437.571	437.510
f_{10}	523.557	524.045	525.718	525.638	525.902	525.823	436.735	436.676	437.571	437.510
Foundation properties: $\rho_F^+ = 1800\,\mathrm{kg/m^3}$, $h_F^+ = 0.1$ m, $k_n^+ = 7.5\times10^7\ \mathrm{N/m^3}$, $G_F^+ = 5\times10^6\ \mathrm{N/m}$, $k_\varphi^+ = k_s^+ = 3.75\times10^7\ \mathrm{N/m^3}$										
Mode [Hz]	$k_n^+ = 0, k_\varphi^+ = 0,$ $k_s^+ = 0, G_F^+ = 0,$ $\rho_F^+ = 0, h_F^+ = 0$		$k_n^+ \neq 0, k_\varphi^+ = 0,$ $k_s^+ = 0, G_F^+ = 0,$ $\rho_F^+ = 0, h_F^+ = 0$		$k_n^+ \neq 0, k_\varphi^+ = 0,$ $k_s^+ = 0, G_F^+ \neq 0,$ $\rho_F^+ = 0, h_F^+ = 0$		$k_n^+ \neq 0, k_\varphi^+ = 0,$ $k_s^+ = 0, G_F^+ \neq 0,$ $\rho_F^+ \neq 0, h_F^+ \neq 0$		$k_n^+ \neq 0, k_\varphi^+ \neq 0,$ $k_s^+ \neq 0, G_F^+ \neq 0,$ $\rho_F^+ \neq 0, h_F^+ \neq 0$	
	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM
f_1	439.664	439.768	445.042	444.884	446.622	446.002	367.911	367.675	373.636	373.453
f_2	447.296	447.842	458.575	459.076	463.802	462.902	386.390	386.755	388.365	388.749
f_3	447.296	447.842	458.575	459.076	463.802	462.902	386.390	386.755	388.365	388.749
f_4	461.762	461.978	469.744	470.056	475.005	474.089	397.332	397.681	399.100	399.403
f_5	461.762	462.406	474.179	474.803	480.284	479.343	402.181	402.691	403.026	403.536
f_6	461.907	462.406	474.179	474.803	480.284	479.343	402.181	402.691	403.026	403.536
f_7	487.146	487.753	499.571	500.162	506.979	506.030	425.495	425.981	425.925	426.412
f_8	487.146	487.753	499.571	500.162	506.979	506.030	425.495	425.981	425.925	426.412
f_9	523.557	524.045	528.406	528.339	530.434	529.901	440.816	440.787	444.754	444.712
f_{10}	523.557	524.045	528.406	528.339	530.434	529.901	440.816	440.787	444.754	444.712

*Control points and weights of the Bézier curve: $\bar{\boldsymbol{x}}_1 = [0.8 \quad 1.3 \quad 1.5 \quad 1.4 \quad 1.2]$, $\bar{\boldsymbol{x}}'_3 = [0 \quad 0.5 \quad 1 \quad 1.5 \quad 2]$, $\boldsymbol{w} = [1 \quad 1 \quad 1 \quad 1 \quad 1]$, Geometric characteristics: $\vartheta_0 = 360^\circ$, $h = 0.1$ m, $R_b = 0$ m, Orthotropic material: $E_1 = 137.9$ GPa, $E_2 = 8.96$ GPa, $G_{12} = G_{13} = 7.1$ GPa, $G_{23} = 6.21$ GPa, $\nu_{12} = 0.3$, $\rho = 1450\,\mathrm{kg/m^3}$.

Table 6. First ten frequencies for a $(0/30)$ CCCC panel resting on Winkler-Pasternak elastic foundation.

Foundation properties: $\rho_F^- = 1800\,\mathrm{kg/m^3}$, $h_F^- = 0.1$ m, $k_n^- = 1.5\times10^7\ \mathrm{N/m^3}$, $G_F^- = 5\times10^5$ N/m, $k_\varphi^- = k_s^- = 0.75\times10^7\ \mathrm{N/m^3}$										
Mode [Hz]	$k_n^- = 0, k_\varphi^- = 0,$ $k_s^- = 0, G_F^- = 0,$ $\rho_F^- = 0, h_F^- = 0$		$k_n^- \neq 0, k_\varphi^- = 0,$ $k_s^- = 0, G_F^- = 0,$ $\rho_F^- = 0, h_F^- = 0$		$k_n^- \neq 0, k_\varphi^- = 0,$ $k_s^- = 0, G_F^- \neq 0,$ $\rho_F^- = 0, h_F^- = 0$		$k_n^- \neq 0, k_\varphi^- = 0,$ $k_s^- = 0, G_F^- \neq 0,$ $\rho_F^- \neq 0, h_F^- \neq 0$		$k_n^- \neq 0, k_\varphi^- \neq 0,$ $k_s^- \neq 0, G_F^- \neq 0,$ $\rho_F^- \neq 0, h_F^- \neq 0$	
	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM
f_1	340.255	340.145	343.424	343.314	344.415	344.305	289.361	289.262	289.649	289.551
f_2	350.272	350.142	353.307	353.178	354.343	354.214	297.893	297.779	298.185	298.072
f_3	383.041	382.642	385.835	385.437	386.912	386.515	325.387	325.053	325.644	325.312
f_4	431.969	431.332	434.468	433.833	435.598	434.964	366.416	365.892	366.635	366.111
f_5	490.418	489.664	492.658	491.905	493.849	493.096	415.329	414.719	415.507	414.898
f_6	550.776	550.109	552.802	552.136	554.056	553.391	465.703	465.187	465.853	465.337
f_7	601.902	601.885	603.847	603.828	605.086	605.066	507.263	507.223	507.379	507.339
f_8	609.556	609.133	611.415	610.992	612.731	612.312	513.997	514.047	514.112	514.162
f_9	609.726	609.801	611.641	611.713	612.928	612.995	514.617	514.308	514.746	514.437
f_{10}	625.460	625.588	627.322	627.449	628.672	628.797	527.445	527.524	527.555	527.635
Foundation properties: $\rho_F^- = 1800\,\mathrm{kg/m^3}$, $h_F^- = 0.1$ m, $k_n^- = 7.5\times10^7\ \mathrm{N/m^3}$, $G_F^- = 5\times10^6$ N/m, $k_\varphi^- = k_s^- = 3.75\times10^7\ \mathrm{N/m^3}$										
Mode [Hz]	$k_n^- = 0, k_\varphi^- = 0,$ $k_s^- = 0, G_F^- = 0,$ $\rho_F^- = 0, h_F^- = 0$		$k_n^- \neq 0, k_\varphi^- = 0,$ $k_s^- = 0, G_F^- = 0,$ $\rho_F^- = 0, h_F^- = 0$		$k_n^- \neq 0, k_\varphi^- = 0,$ $k_s^- = 0, G_F^- \neq 0,$ $\rho_F^- = 0, h_F^- = 0$		$k_n^- \neq 0, k_\varphi^- = 0,$ $k_s^- = 0, G_F^- \neq 0,$ $\rho_F^- \neq 0, h_F^- \neq 0$		$k_n^- \neq 0, k_\varphi^- \neq 0,$ $k_s^- \neq 0, G_F^- \neq 0,$ $\rho_F^- \neq 0, h_F^- \neq 0$	
	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM
f_1	340.255	340.145	355.807	355.698	365.230	365.119	306.853	306.754	308.218	308.122
f_2	350.272	350.142	365.186	365.055	375.044	374.915	315.303	315.190	316.692	316.583
f_3	383.041	382.642	396.807	396.411	407.115	406.728	342.384	342.060	343.617	343.298
f_4	431.969	431.332	444.315	443.684	455.191	454.573	382.908	382.397	383.964	383.457
f_5	490.418	489.664	501.513	500.763	513.035	512.296	431.474	430.876	432.341	431.748
f_6	550.776	550.109	560.830	560.164	573.021	572.361	481.654	481.142	482.386	481.877
f_7	601.902	601.885	611.560	611.534	623.660	623.625	522.837	522.784	523.401	523.353
f_8	609.556	609.133	618.789	618.365	631.583	631.223	529.787	529.822	530.350	530.390
f_9	609.726	609.801	619.240	619.302	631.844	631.828	530.535	530.220	531.161	530.847
f_{10}	625.460	625.588	634.715	634.834	647.906	648.011	543.587	543.649	544.126	544.192

*Control points and weights of the Bézier curve: $\bar{\mathbf{x}}_1 = [0\ \ 0.5\ \ 1\ \ 1.5\ \ 2]$, $\bar{\mathbf{x}}_3' = [0.8\ \ 1.3\ \ 1.5\ \ 1.3\ \ 0.8]$, $\mathbf{w} = [1\ \ 1\ \ 1\ \ 1\ \ 1]$, Geometric characteristics: $\vartheta_0 = 90^\circ$, $h = 0.1$ m, $R_b = 1$ m, Orthotropic material: $E_1 = 137.9$ GPa, $E_2 = 8.96$ GPa, $G_{12} = G_{13} = 7.1$ GPa, $G_{23} = 6.21$ GPa, $\nu_{12} = 0.3$, $\rho = 1450\,\mathrm{kg/m^3}$.

Table 7. First ten frequencies for a $(30/60)$ FC shell resting on Winkler-Pasternak elastic foundation.

Mode [Hz]	$k_n^- = 0, k_\varphi^- = 0,$ $k_s^- = 0, G_F^- = 0,$ $\rho_F^- = 0, h_F^- = 0$		$k_n^- \neq 0, k_\varphi^- = 0,$ $k_s^- = 0, G_F^- = 0,$ $\rho_F^- = 0, h_F^- = 0$		$k_n^- \neq 0, k_\varphi^- = 0,$ $k_s^- = 0, G_F^- \neq 0,$ $\rho_F^- = 0, h_F^- = 0$		$k_n^- \neq 0, k_\varphi^- = 0,$ $k_s^- = 0, G_F^- \neq 0,$ $\rho_F^- \neq 0, h_F^- \neq 0$		$k_n^- \neq 0, k_\varphi^- \neq 0,$ $k_s^- \neq 0, G_F^- \neq 0,$ $\rho_F^- \neq 0, h_F^- \neq 0$	
Foundation properties: $\rho_F^- = 1800\,\mathrm{kg/m^3}$, $h_F^- = 0.1$ m, $k_n^- = 1.5\times10^7\,\mathrm{N/m^3}$, $G_F^- = 5\times10^5\,\mathrm{N/m}$, $k_\varphi^- = k_s^- = 0.75\times10^7\,\mathrm{N/m^3}$										
	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM
f_1	72.894	73.118	85.036	85.215	85.527	85.705	71.828	71.978	73.601	73.752
f_2	80.378	80.658	91.421	91.653	92.019	92.246	77.392	77.581	79.033	79.225
f_3	80.378	80.658	91.421	91.653	92.019	92.246	77.392	77.581	79.033	79.225
f_4	100.235	101.287	109.478	110.428	110.279	111.221	92.985	93.778	94.201	94.991
f_5	100.235	101.287	109.478	110.428	110.279	111.221	92.985	93.778	94.201	94.991
f_6	120.959	122.408	129.744	131.116	130.877	132.261	110.158	111.325	110.812	111.964
f_7	120.959	122.408	129.744	131.116	130.877	132.261	110.158	111.325	110.812	111.964
f_8	139.742	140.410	148.060	148.701	149.746	150.406	125.669	126.227	126.043	126.595
f_9	139.742	140.410	148.060	148.701	149.746	150.406	125.669	126.227	126.043	126.595
f_{10}	166.126	166.471	173.369	173.699	175.529	175.867	147.128	147.416	147.395	147.681
Foundation properties: $\rho_F^- = 1800\,\mathrm{kg/m^3}$, $h_F^- = 0.1$ m, $k_n^- = 7.5\times10^7\,\mathrm{N/m^3}$, $G_F^- = 5\times10^6\,\mathrm{N/m}$, $k_\varphi^- = k_s^- = 3.75\times10^7\,\mathrm{N/m^3}$										
Mode [Hz]	$k_n^- = 0, k_\varphi^- = 0,$ $k_s^- = 0, G_F^- = 0,$ $\rho_F^- = 0, h_F^- = 0$		$k_n^- \neq 0, k_\varphi^- = 0,$ $k_s^- = 0, G_F^- = 0,$ $\rho_F^- = 0, h_F^- = 0$		$k_n^- \neq 0, k_\varphi^- = 0,$ $k_s^- = 0, G_F^- \neq 0,$ $\rho_F^- = 0, h_F^- = 0$		$k_n^- \neq 0, k_\varphi^- = 0,$ $k_s^- = 0, G_F^- \neq 0,$ $\rho_F^- \neq 0, h_F^- \neq 0$		$k_n^- \neq 0, k_\varphi^- \neq 0,$ $k_s^- \neq 0, G_F^- \neq 0,$ $\rho_F^- \neq 0, h_F^- \neq 0$	
	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM	GDQ-GST	GDQ-RM
f_1	72.894	73.118	121.690	121.783	124.987	125.077	104.927	105.003	111.193	111.279
f_2	80.378	80.658	125.891	126.017	130.041	130.134	109.337	109.414	115.358	115.454
f_3	80.378	80.658	125.891	126.017	130.041	130.134	109.337	109.414	115.358	115.454
f_4	100.235	101.287	140.166	140.861	146.045	146.686	123.125	123.662	127.955	128.505
f_5	100.235	101.287	140.166	140.861	146.045	146.686	123.125	123.662	127.955	128.505
f_6	120.959	122.408	159.875	161.052	168.401	169.659	141.751	142.810	144.550	145.569
f_7	120.959	122.408	159.875	161.052	168.401	169.659	141.751	142.810	144.550	145.569
f_8	139.742	140.410	177.411	177.983	190.552	191.285	159.855	160.471	161.472	162.059
f_9	139.742	140.410	177.411	177.983	190.552	191.285	159.855	160.471	161.472	162.059
f_{10}	166.126	166.471	199.712	200.002	207.398	207.673	173.861	174.088	175.326	175.575

*Control points and weights of the Bézier curve: $\bar{\boldsymbol{x}}_1 = \begin{bmatrix} 0 & 0.3 & 1 & 1.5 & 2 \end{bmatrix}$, $\bar{\boldsymbol{x}}_3' = \begin{bmatrix} 2 & 1.2 & 0.85 & 0.75 & 0.7 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$, Geometric characteristics: $\vartheta_0 = 360^\circ$, $h = 0.1$ m, $R_b = 2$ m, Orthotropic material: $E_1 = 137.9$ GPa, $E_2 = 8.96$ GPa, $G_{12} = G_{13} = 7.1$ GPa, $G_{23} = 6.21$ GPa, $\nu_{12} = 0.3$, $\rho = 1450\,\mathrm{kg/m^3}$.

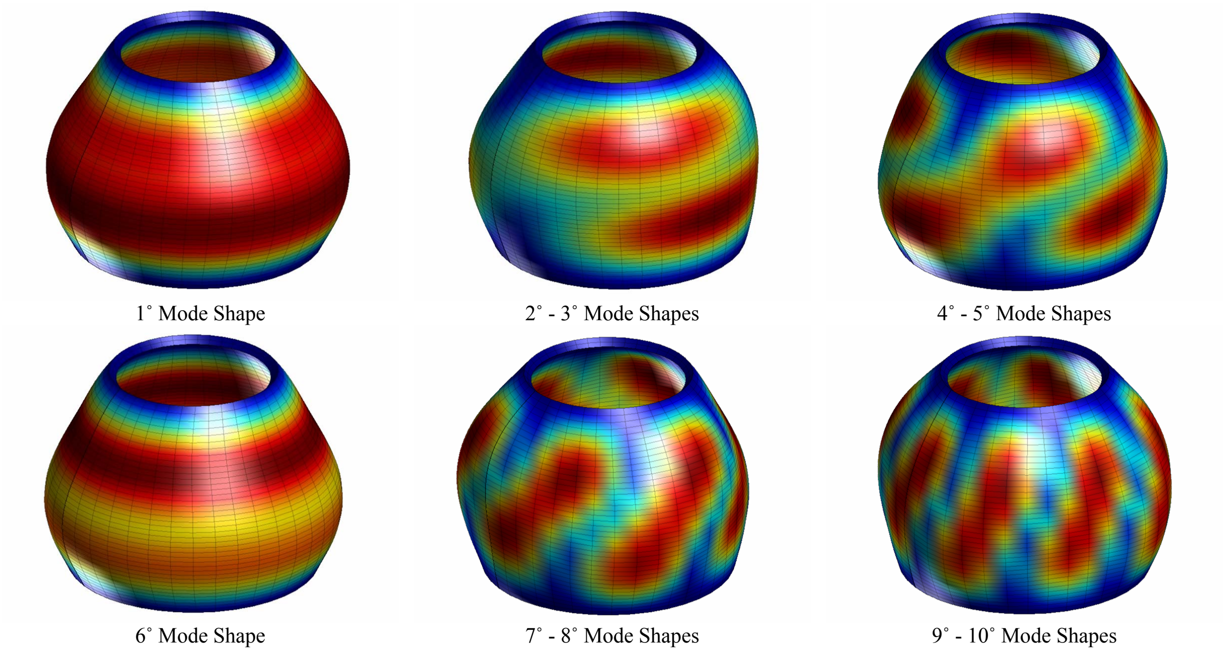

Figure 6. Mode shapes for the CC shell of Table 5.

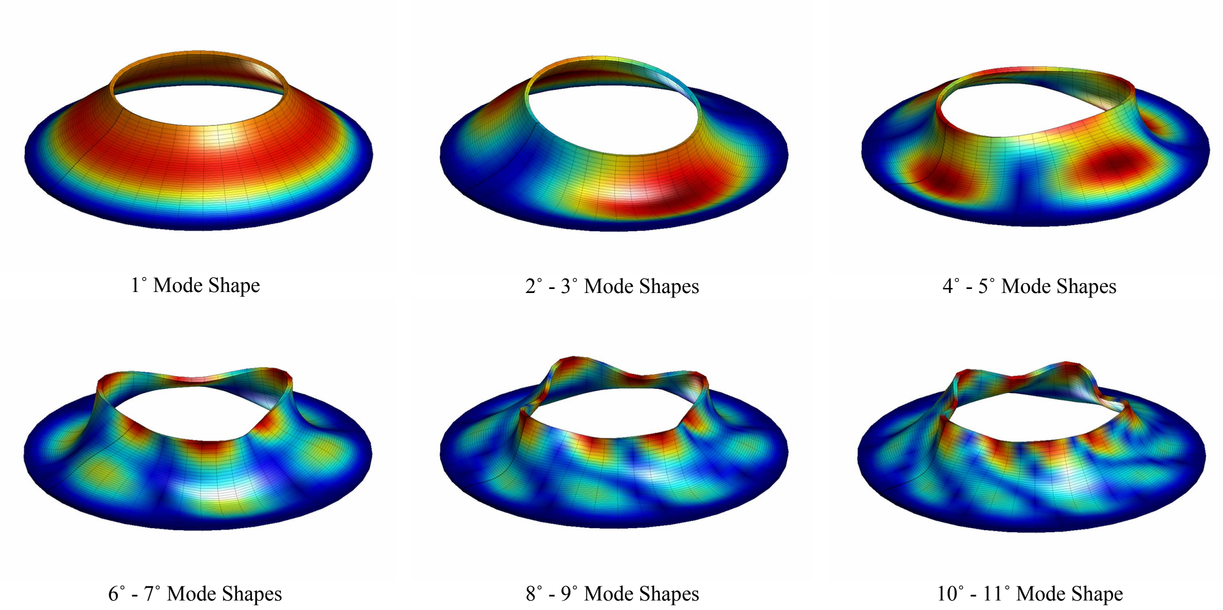

Figure 7. Mode shapes for the FC shell of Table 7.

Tornabene *et al*. [69], respectively. In both cases the discrepancy between results obtained is closely zero.

5. Conclusion

The static and free vibration analyses of laminated doubly-curved shells and panels of revolution resting on Winkler-Pasternak elastic foundations have been presented using the GDQ method. All the effects of the foundation, except the damping, are separately introduced. New results are presented in order to investigate the effects of the Winkler modulus, the Pasternak modulus and the inertia of the elastic foundation on the behavior of laminated shells of revolution. Simple Rational Bézier curves are used to mathematically describe the shape of the curved shell. Various lamination schemes with different layers have been considered. A general FSDT shell theory has been obtained considering the curvature effect. The fundamental equilibrium equations have been discretized with the GDQ method giving a standard linear problem for the static analysis and a standard linear eigenvalue problem for the dynamic analysis. Numerical solutions have been compared with the ones obtained using commercial programs. The comparisons conducted with FEM codes confirm how the GDQ simple numerical method provides accurate and computationally low cost results for all the structures considered. Furthermore, discretizing and programming procedures are quite easy. The GDQ results show to be precise and reliable. The numerical tests demonstrate and confirm the favorable precision of the Generalized Differential Quadrature Method.

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Appendix

The engineering elastic stiffnesses $\overline{A}_{ijm}^{(q)}$ which depend on curvatures for a doubly-curved shell of revolution are defined as it follows:

$$\begin{aligned}\overline{A}_{ij\varphi}^{(q)} &= \sum_{k=1}^{l}\int_{\zeta_k}^{\zeta_{k+1}} \overline{Q}_{ij}^{(k)}\zeta^q \frac{1+\zeta/R_s}{1+\zeta/R_\varphi}\mathrm{d}\zeta \\ \overline{A}_{ijs}^{(q)} &= \sum_{k=1}^{l}\int_{\zeta_k}^{\zeta_{k+1}} \overline{Q}_{ij}^{(k)}\zeta^q \frac{1+\zeta/R_\varphi}{1+\zeta/R_s}\mathrm{d}\zeta\end{aligned} \tag{44}$$

Using the Taylor geometric series expansion [7,16-18, 85], it is possible to introduce the following approximations:

$$\begin{aligned}\frac{1+\zeta/R_s}{1+\zeta/R_\varphi} &\cong 1+\left(\frac{1}{R_s}-\frac{1}{R_\varphi}\right)\sum_{p=1}^{r}(-1)^{p+1}\frac{\zeta^p}{R_\varphi^{p-1}}+o\left(\zeta^r\right) \\ \frac{1+\zeta/R_s}{1+\zeta/R_\varphi} &\cong 1+\left(\frac{1}{R_s}-\frac{1}{R_\varphi}\right)\sum_{p=1}^{r}(-1)^{p+2}\frac{\zeta^p}{R_s^{p-1}}+o\left(\zeta^r\right)\end{aligned} \tag{45}$$

where r is the maximum order of the series expansions. By neglecting the series terms (45) with order higher than r and by introducing the relations (45) into the integrals (44), it is obtained:

$$\begin{aligned}\overline{A}_{ij\varphi}^{(q)} &\cong \sum_{k=1}^{l}\int_{\zeta_k}^{\zeta_{k+1}} \overline{Q}_{ij}^{(k)}\zeta^q \left(1+\left(\frac{1}{R_s}-\frac{1}{R_\varphi}\right)\sum_{p=1}^{r}(-1)^{p+1}\frac{\zeta^p}{R_\varphi^{p-1}}\right)\mathrm{d}\zeta \\ \overline{A}_{ijs}^{(q)} &\cong \sum_{k=1}^{l}\int_{\zeta_k}^{\zeta_{k+1}} \overline{Q}_{ij}^{(k)}\zeta^q \left(1+\left(\frac{1}{R_s}-\frac{1}{R_\varphi}\right)\sum_{p=1}^{r}(-1)^{p+2}\frac{\zeta^p}{R_s^{p-1}}\right)\mathrm{d}\zeta\end{aligned} \tag{46}$$

Thus, the engineering elastic stiffnesses $\overline{A}_{ijm}^{(q)}$ can be evaluated in the following manner:

$$\begin{aligned}\overline{A}_{ij\varphi}^{(q)} &\cong \sum_{k=1}^{l}\left(\int_{\zeta_k}^{\zeta_{k+1}} \overline{Q}_{ij}^{(k)}\zeta^q\mathrm{d}\zeta+\left(\frac{1}{R_s}-\frac{1}{R_\varphi}\right)\sum_{p=1}^{r}\frac{(-1)^{p+1}}{R_\varphi^{p-1}}\int_{\zeta_k}^{\zeta_{k+1}}\overline{Q}_{ij}^{(k)}\zeta^{q+p}\mathrm{d}\zeta\right) \\ \overline{A}_{ijs}^{(q)} &\cong \sum_{k=1}^{l}\left(\int_{\zeta_k}^{\zeta_{k+1}} \overline{Q}_{ij}^{(k)}\zeta^q\mathrm{d}\zeta+\left(\frac{1}{R_s}-\frac{1}{R_\varphi}\right)\sum_{p=1}^{r}\frac{(-1)^{p+2}}{R_s^{p-1}}\int_{\zeta_k}^{\zeta_{k+1}}\overline{Q}_{ij}^{(k)}\zeta^{q+p}\mathrm{d}\zeta\right)\end{aligned} \tag{47}$$

The same results can be exactly obtained using the expressions proposed by Qatu [16], by approximating the logarithmic function using a Taylor series expansion. Furthermore, the relations (47) can be defined in the form:

$$\overline{A}_{ij\varphi}^{(q)} \cong A_{ij}^{(q)}+\left(\frac{1}{R_s}-\frac{1}{R_\varphi}\right)\sum_{p=1}^{r}(-1)^{p+1}\frac{A_{ij}^{(q+p)}}{R_\varphi^{p-1}},\ \overline{A}_{ijs}^{(q)} \cong A_{ij}^{(q)}+\left(\frac{1}{R_s}-\frac{1}{R_\varphi}\right)\sum_{p=1}^{r}(-1)^{p+2}\frac{A_{ij}^{(q+p)}}{R_s^{p-1}} \tag{48}$$

where:

$$A_{ij}^{(n)} = \sum_{k=1}^{l}\int_{\zeta_k}^{\zeta_{k+1}}\overline{Q}_{ij}^{(k)}\zeta^n\mathrm{d}\zeta \tag{49}$$

Finally, the exact integration of the expressions (47) assume the following aspect:

$$\begin{aligned}\overline{A}_{ij\varphi}^{(q)} &\cong \sum_{k=1}^{l}\overline{Q}_{ij}^{(k)}\left(\frac{\zeta_{k+1}^{q+1}-\zeta_k^{q+1}}{q+1}+\left(\frac{1}{R_s}-\frac{1}{R_\varphi}\right)\sum_{p=1}^{r}\frac{(-1)^{p+1}}{R_\varphi^{p-1}}\frac{\zeta_{k+1}^{q+p+1}-\zeta_k^{q+p+1}}{q+p+1}\right) \\ \overline{A}_{ijs}^{(q)} &\cong \sum_{k=1}^{l}\overline{Q}_{ij}^{(k)}\left(\frac{\zeta_{k+1}^{q+1}-\zeta_k^{q+1}}{q+1}+\left(\frac{1}{R_s}-\frac{1}{R_\varphi}\right)\sum_{p=1}^{r}\frac{(-1)^{p+2}}{R_s^{p-1}}\frac{\zeta_{k+1}^{q+p+1}-\zeta_k^{q+p+1}}{q+p+1}\right)\end{aligned} \tag{50}$$

whenever the reduced elastic constants $\overline{Q}_{ij}^{(k)}$ do not depend on ζ co-ordinate. The engineering elastic stiffness $\overline{A}_{ijm}^{(q)}$ definitions proposed in the present work represent a generalization of the ones proposed in [7,16-18, 77-79,85].