Hindawi
Advances in High Energy Physics
Volume 2020, Article ID 3967605, 8 pages
https://doi.org/10.1155/2020/3967605

*Research Article*

# New Partial Symmetries from Group Algebras for Lepton Mixing

**Shu-Jun Rong**

*College of Science, Guilin University of Technology, Guilin, Guangxi 541004, China*

Correspondence should be addressed to Shu-Jun Rong; rongshj@glut.edu.cn

Received 13 November 2019; Revised 6 January 2020; Accepted 29 January 2020; Published 8 February 2020

Academic Editor: Edward Sarkisyan-Grinbaum



Recent stringent experiment data of neutrino oscillations induces partial symmetries such as $Z_2$ and $Z_2 \times \mathrm{CP}$ to derive lepton mixing patterns. New partial symmetries expressed with elements of group algebras are studied. A specific lepton mixing pattern could correspond to a set of equivalent elements of a group algebra. The transformation which interchanges the elements could express a residual CP symmetry. Lepton mixing matrices from $S_3$ group algebras are of the trimaximal form with the $\mu-\tau$ reflection symmetry. Accordingly, elements of $S_3$ group algebras are equivalent to $Z_2 \times \mathrm{CP}$. Comments on $S_4$ group algebras are given. The predictions of $Z_2 \times \mathrm{CP}$ broken from the group $S_4$ with the generalized CP symmetry are also obtained from elements of $S_4$ group algebras.

## 1. Introduction

Discoveries of neutrino oscillation [1–3] opened a window to physics beyond the standard model. In order to explain possible patterns of lepton mixing parameters, discrete flavor symmetries were extensively investigated in recent decades [4–21]. The general route on this approach is as follows. First, suppose that the Lagrangian of leptons is invariant under actions of some finite group $G_f$.

After symmetry breaking from vacuum expectation values of scalar multiplets, $G_f$ is reduced to $G_e$ in the charged lepton section and $G_\nu$ in the neutrino section. Accordingly, the mass matrix of charged leptons is invariant under some unitary transformation, i.e.,

$$X_e^+ M_e^+ M_e X_e = M_e^+ M_e. \tag{1}$$

So we have

$$\begin{aligned} U_e^+ M_e^+ M_e U_e &= \mathrm{diag}\left(m_e^2, m_\mu^2, m_\tau^2\right), \\ U_e^+ X_e U_e &= \mathrm{diag}\left(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}\right). \end{aligned} \tag{2}$$

The counterparts for Dirac neutrinos are written as

$$\begin{aligned} X_\nu^+ M_\nu^+ M_\nu X_\nu &= M_\nu^+ M_\nu, \\ U_\nu^+ M_\nu^+ M_\nu U_\nu &= \mathrm{diag}\left(m_1^2, m_2^2, m_3^2\right), \\ U_\nu^+ X_\nu U_\nu &= \mathrm{diag}\left(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}\right). \end{aligned} \tag{3}$$

For Majorana neutrinos, they read

$$\begin{aligned} X_\nu^T M_\nu X_\nu &= M_\nu, \\ U_\nu^T M_\nu U_\nu &= \mathrm{diag}\left(m_1, m_2, m_3\right), \\ U_\nu^+ X_\nu U_\nu &= \mathrm{diag}\left(\pm1, \pm1, \pm1\right). \end{aligned} \tag{4}$$

So residual symmetries $X_e$ and $X_\nu$ can determine the lepton mixing matrix $U_{\mathrm{PMNS}} \equiv U_e^+ U_\nu$ up to permutations of rows or columns.

However, mixing patterns based on small flavor groups cannot accommodate new stringent experiment data, especially the nonzero mixing angle $\theta_{13}$. Although some large groups could give a viable $\theta_{13}$, the Dirac

CP-violating phase from them is trivial [22]. In order to alleviate the tension between predictions of flavor groups and experiment constraints, one can resort to partial symmetries. Namely, the lepton mixing matrix is partially determined by symmetries such as $Z_2$ [23–25] and $Z_2 \times \mathrm{CP}$ [26–43]. Here CP denotes a generalized CP transformation (GCP). For $Z_2$ symmetries, an unfixed unitary rotation is contained in the mixing matrix. Even so, they may predict some mixing angle, Dirac CP phase, or correlation of them. If the residual symmetry is $(Z_{2e} \times Z_{2e}, Z_{2\nu} \times \mathrm{CP}_\nu)$ or $(Z_{ne}, Z_{2\nu} \times CP_\nu)$ with $n \geq 3$, the Dirac CP phase would be trivial or maximal in the case that the residual flavor group is from small groups $S_4$ and $A_5$ [30, 32, 39]. Here, the symmetries of the charged lepton sector and those of neutrinos are marked with the subscripts $e$ and $\nu$, respectively. To obtain a more general CP phase, one can choose the residual symmetry $(Z_{2e} \times \mathrm{CP}_e, Z_{2\nu} \times \mathrm{CP}_\nu)$ [44, 45]. Then, the lepton mixing matrix contains two angle parameters to constrain by experiment data.

In this paper, we explore a new construct to describe partial symmetries which was proposed recently in Ref. [46]. The partial symmetry is expressed by an element of a group algebra. According to Ref. [47], a group algebra $K[G]$ is the set of all linear combinations of elements of the group $G$ with coefficients in the field $K$. A general element of $K[G]$ is denoted as

$$\sum_{g\in G} a_g g. \tag{5}$$

$K[G]$ is an algebra over $K$ with the addition and multiplication defined, respectively, as

$$\begin{gathered}\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} \left(a_g + b_g\right) g, \\ \left(\sum_{g\in G} a_g g\right)\left(\sum_{h\in G} b_h h\right) = \sum_{g\in G, h\in G} \left(a_g b_h\right) g \cdot h,\end{gathered} \tag{6}$$

where the operation "·" denotes the multiplication of group elements. The product by a scalar is defined as

$$a\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} \left(a a_g\right) \mathrm{g}. \tag{7}$$

From the above definitions, we can see that a group algebra describes the superposition of symmetries expressed by group elements. Similar to the residual symmetry $Z_2 \times \mathrm{CP}$, the elements of a group algebra with continuous superposition coefficients may also describe partial symmetries of leptons. They may be used to predict the lepton mixing pattern. For simplicity, we consider the group algebra constructed by two group elements in this paper. Namely, the residual symmetry is expressed as

$$X_{e,\nu} = x_{1e,\nu} A_{1e,\nu} + x_{2e,\nu} A_{2e,\nu}, \tag{8}$$

where $A_{1e,\nu}$ and $A_{2e,\nu}$ are elements of a small group. Through equivalent transformations, the superposition coefficients are dependent on a real parameter in a special parametrization. So we can obtain clear relations between mixing parameters and the adjustable coefficient. In spite of the economy of the structure, $X_{e,\nu}$ seems strange. It is not a group element in general. The choice of $A_i$ seems random. To realize the characteristic of the novel construct, we study a minimal case with the $S_3$ group algebra. We find that $X$ in the $S_3$ group algebra is equivalent to the symmetry $Z_2 \times \mathrm{CP}$ in the case of Dirac neutrinos. Furthermore, the maximal or trivial Dirac CP phase could be obtained from $X$ in the $S_4$ group algebra. Although we cannot prove that the equivalence holds for $X$ in a general algebra, we may have more choices in the realization of partial symmetries.

This paper is organised as follows. In Section 2, we show an economical realization of group algebras. In Section 3, we study a minimal case with an $S_3$ group algebra. Finally, we give a conclusion.

## 2. Realization of a Group Algebra

An element of a group algebra is constructed by the superposition of elements of a group. Here, we consider the elements of group algebras obtained from two group elements. We note that the representation matrix of $X$ is not unitary in general even if the representation of the group elements is unitary. In order to keep the representation of $X$ unitary, we set extra constraints on coefficients and group elements, namely,

$$\begin{cases} \left(|x_1|^2 + |x_2|^2\right) I + x_1 x_2^* A_1 A_2^+ + x_2 x_1^* A_2 A_1^+ = I, \\ \left(|x_1|^2 + |x_2|^2\right) I + x_2 x_1^* A_1^+ A_2 + x_1 x_2^* A_2^+ A_1 = I, \end{cases} \tag{9}$$

where the signal "∗" denotes the complex conjugation. An economical solution to the constraint equations is

$$\begin{cases} |x_1|^2 + |x_2|^2 = 1, \\ e^{i\alpha} A_1 A_2^+ + e^{-i\alpha} \mathrm{A}_2 A_1^+ = O, \\ e^{-i\alpha} A_1^+ A_2 + e^{i\alpha} A_2^+ A_1 = O, \end{cases} \tag{10}$$

where $\alpha$ is the phase of the term $x_1 x_2^*$ and $O$ is the zero matrix. Up to a global phase, by a redefinition of the matrix $A_1$ or $A_2$, $X$ can be parameterized as [46]

$$X(\theta) = \cos\theta A_1 + i\sin\theta A_2, \tag{11}$$

where $i$ is the imaginary factor and $A_1$ and $A_2$ satisfy the constraints

$$\begin{aligned} A_1A_2^+ &= A_2A_1^+, \\ A_1^+A_2 &= A_2^+A_1. \end{aligned} \tag{12}$$

So $A_1A_2^+$ and $A_1^+A_2$ are generators of $Z_2$ groups. $X$ can be rewritten as $X = A_1e^{i\theta B}$ with $B = A_1^+A_2$, $B^2 = I$.

Let us make some necessary comments here:

(a) For Majorana neutrinos, the residual symmetry is $Z_2 \times Z_2$. It can be broken to the partial symmetry $Z_2$. $X$ depends on a continuous parameter $\theta$. It is not a $Z_2$ symmetry in general. So $X$ is used for the description of residual symmetries of charged leptons and Dirac neutrinos

(b) With a special choice of group elements $A_i$ and the parameter $\theta$, $X$ could become a generator of a large cyclic group. An example is given in Ref. [46]

(c) The mixing matrix from $X(\theta)$ is dependent on a parameter $\theta$. Furthermore, $X(\theta)$ is equivalent to $Z_2 \times \mathrm{CP}$ in the case of $S_3$ group algebras. This interesting observation still holds for some elements of $S_4$ group algebras

(d) Although $X$ is dependent on the parameter $\theta$, some mixing angle or CP phase may be independent of $\theta$. We may separate impacts of discrete group elements and $\theta$ in special cases

## 3. A Minimal Case for $S_3$ Group Algebra

For illustration, we consider a minimal case that the group algebra is constructed by elements of the group $S_3$. Although the 3-dimensional representation of $S_3$ group algebras is reducible, it can be viewed as the special case of $S_4$ group algebras. In this section, we first consider the special case that the mass matrix of charged leptons is diagonal. So the lepton mixing matrix is just dependent on the residual symmetry $X_\nu$. Then, we show equivalence of elements of $S_3$ group algebras and the residual symmetry $Z_2 \times \mathrm{CP}$. Comments on $S_4$ group algebras are also made. Finally, we discuss general residual symmetries of the charged lepton sector.

### *3.1. Mixing Patterns from $S_3$ Group Algebra in the Case of the Diagonal Mass Matrix $M_e^+M_e$.*

The 3-dimensional reducible representation of the group $S_3$ is expressed as

$$\begin{aligned} I &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ S_{12} &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ S_{13} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ S_{23} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ S_{123} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ S_{132} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{13}$$

According to the unitary conditions of Equation (12), viable nontrivial realizations of $X_\nu$ are listed as

$$\begin{aligned} X_{1\nu} &\equiv S_{23}e^{i\theta S_{12}}, \\ X_{2\nu} &\equiv S_{23}e^{i\theta S_{13}}, \\ X_{3\nu} &\equiv S_{12}e^{i\theta S_{13}}, \\ X_{4\nu} &\equiv S_{12}e^{i\theta S_{23}}, \\ X_{5\nu} &\equiv S_{13}e^{i\theta S_{12}}, \\ X_{6\nu} &\equiv S_{13}e^{i\theta S_{23}}, \\ X_{7\nu} &\equiv S_{123}e^{i\theta S_{12}}, \\ X_{8\nu} &\equiv S_{123}e^{i\theta S_{23}}, \\ X_{9\nu} &\equiv S_{123}e^{i\theta S_{13}}, \\ X_{10\nu} &\equiv S_{132}e^{i\theta S_{12}}, \\ X_{11\nu} &\equiv S_{132}e^{i\theta S_{23}}, \\ X_{12\nu} &\equiv S_{132}e^{i\theta S_{13}}. \end{aligned} \tag{14}$$

All theses $X_\nu$ correspond to the same lepton mixing matrix up to permutations of rows, columns, or trivial phases. We consider $X_{1\nu}$ as a representative, whose expression is

$$X_{1\nu} \equiv \begin{pmatrix} \cos\theta & i\sin\theta & 0 \\ 0 & 0 & e^{i\theta} \\ i\sin\theta & \cos\theta & 0 \end{pmatrix}. \quad (15)$$

It is diagonalized as

$$U_\nu^+ X_{1\nu} U_\nu = \mathrm{diag}\left(e^{i\theta_1}, e^{i\theta}, e^{i\theta_2}\right), \quad (16)$$

where $e^{i\theta_1} \equiv \sqrt{1-s^2/4} - \mathrm{i}s/2$, $e^{i\theta_2} \equiv -\sqrt{1-s^2/4} - \mathrm{i}s/2$, $s \equiv \sin\theta$. The matrix $U_\nu$ reads

$$U_\nu = \begin{pmatrix} \dfrac{e^{i\theta_1} - ce^{i(\theta-\theta_1)}}{\mathrm{i}s\sqrt{N_1}} & \dfrac{1}{\sqrt{3}} & \dfrac{e^{i\theta_2} - ce^{i(\theta-\theta_2)}}{\mathrm{i}s\sqrt{N_2}} \\ \dfrac{1}{\sqrt{N_1}} & \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{N_2}} \\ \dfrac{e^{i(\theta-\theta_1)}}{\sqrt{N_1}} & \dfrac{1}{\sqrt{3}} & \dfrac{e^{i(\theta-\theta_2)}}{\sqrt{N_2}} \end{pmatrix}, \quad (17)$$

where $c \equiv \cos\theta$, $N_j \equiv 2 + (1 + c^2 - 2c\cos(\theta - 2\theta_j))/s^2$, $j = 1, 2$. It is of trimaximal form with the $\mu-\tau$ reflection symmetry [27, 48–50], i.e., $U_{\alpha 2} = 1/\sqrt{3}$ with $\alpha = e, \mu, \tau$ and $|U_{\mu j}| = |U_{\tau j}|$ with $j = 1, 2, 3$. The lepton mixing matrix $U_{\mathrm{PMNS}}$ is equal to $U_\nu$ up to permutations of rows or columns. Given the recent global fit data of neutrino oscillations[51], viable mixing matrices are

$$U_1 \equiv U_\nu,\ U_2 \equiv S_{23}U_1,\ U_3 \equiv U_1 S_{13},\ U_4 \equiv U_2 S_{13}. \quad (18)$$

Note that $U_3(\theta) = U_1(\theta+\pi)$ and $U_4(\theta) = U_2(\theta+\pi)$. Furthermore, according to the standard parametrization [52]

$$U_{\mathrm{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\mathrm{CP}}} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta_{\mathrm{CP}}} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta_{\mathrm{CP}}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta_{\mathrm{CP}}} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta_{\mathrm{CP}}} & c_{13}c_{23} \end{pmatrix} \cdot \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (19)$$

where $s_{ij} \equiv \sin\theta_{ij}$ and $c_{ij} \equiv \cos\theta_{ij}$, $\delta_{\mathrm{CP}}$ is the Dirac CP-violating phase, $\alpha_1$ and $\alpha_2$ are Majorana phases, and $U_1$ and $U_2$ are interchanged through the following transformation: $\theta_{23} \to \pi/2 - \theta_{23}$ and $\delta_{\mathrm{CP}} \to \delta_{\mathrm{CP}} + \pi$. So without loss of generality, we can just consider $U_1$. Lepton mixing angles and the Dirac CP phase are listed as

$$\begin{aligned} \sin^2\theta_{13} &= \frac{1 + c^2 - 2c\cos(\theta - 2\theta_2)}{2 + s^2 - 2c\cos(\theta - 2\theta_2)}, \\ \sin^2\theta_{12} &= \frac{1}{3\cos^2\theta_{13}}, \\ \sin^2\theta_{23} &= \frac{1}{2}, \\ \delta_{\mathrm{CP}} &= -\,\mathrm{sign}\,(s)\frac{\pi}{2}, \end{aligned} \quad (20)$$

where $s \neq 0$. Dependence of $\sin^2\theta_{13}$ and $\sin^2\theta_{12}$ on the variable $\theta$ is shown in Figure 1. From the figure, we can see that $\sin^2\theta_{12}$ is a slowly varying function of the parameter $\theta$. So the parameter space of $\theta$ is mainly constrained by $\sin^2\theta_{13}$. According to the function $\chi^2$ defined as

$$\chi^2 = \sum_{ij=13,23,12} \left( \frac{\sin^2\theta_{ij} - (\sin^2\theta_{ij})_{\mathrm{exp}}}{\sigma_{ij}} \right)^2, \quad (21)$$

where $(\sin^2\theta_{ij})_{\mathrm{exp}}$ are best global fit values from Ref. [51] and $\sigma_{ij}$ are $1\sigma$ uncertainties; best fit data of $\theta$, $\sin^2\theta_{ij}$, and $\delta_{\mathrm{CP}}$ are listed in Table 1. They are in the $3\sigma$ ranges of the global fit data.

*3.2. Equivalence of Elements of $S_3$ Group Algebras and $Z_2 \times CP$.* The neutrino mass matrix $M_\nu^+ M_\nu$ which is invariant under the action of $X_{1\nu}$ is of the form

$$M_\nu^+ M_\nu = \begin{pmatrix} m_{ee} & m_{e\mu} & m_{e\mu}^* \\ m_{e\mu}^* & m_{\tau\tau} & m_{e\mu} + m_{ee} - m_{\tau\tau} \\ m_{e\mu} & m_{e\mu}^* + m_{ee} - m_{\tau\tau} & m_{\tau\tau} \end{pmatrix}, \quad (22)$$

where $m_{ee}$ and $m_{\tau\tau}$ are real and $\mathrm{Im}\,(m_{e\mu}) = (1/2)(m_{ee} - m_{\tau\tau})\tan\theta$. Obviously, $M_\nu^+ M_\nu$ follows the residual symmetry $Z_2 \times \mathrm{CP}$, i.e.,

$$\begin{aligned} C_{\mathrm{Magic}}^+ (M_\nu^+ M_\nu) C_{\mathrm{Magic}} &= M_\nu^+ M_\nu, \\ S_{23}^+ (M_\nu^+ M_\nu) S_{23} &= (M_\nu^+ M_\nu)^*, \end{aligned} \quad (23)$$

where

$$C_{\mathrm{Magic}} \equiv \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix},\ C_{\mathrm{Magic}}^2 = I. \quad (24)$$

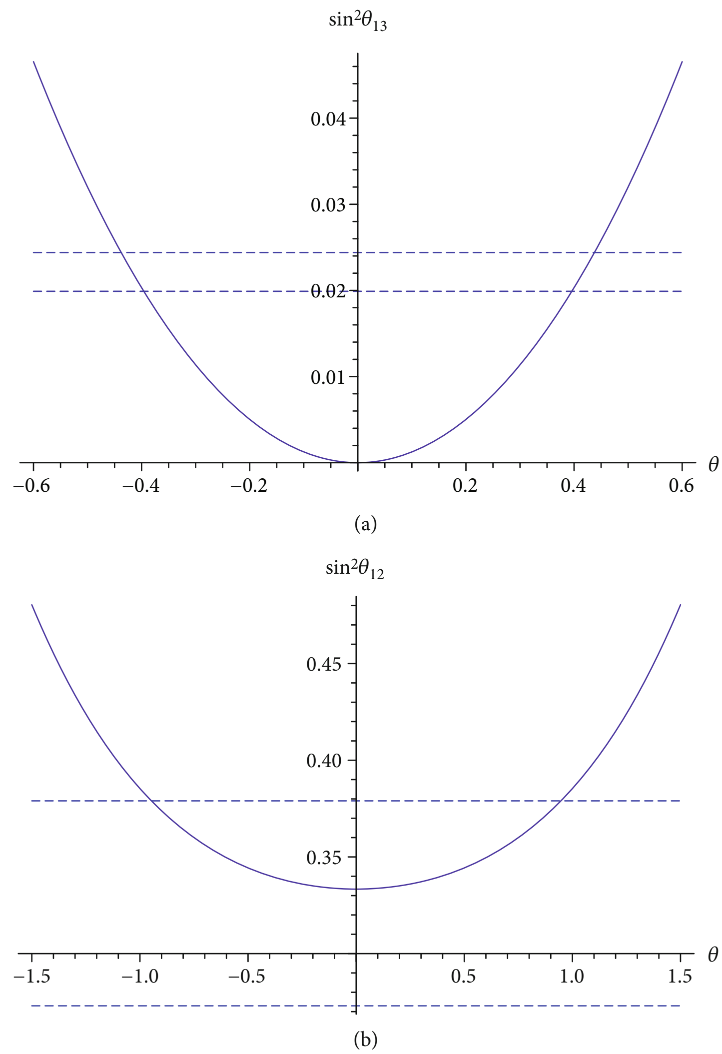


FIGURE 1: Dependence of functions $\sin^2\theta_{13}$ and $\sin^2\theta_{12}$ on the variable $\theta$. Two dashed lines in (a) and (b) label the $3\sigma$ range of the mixing angle from the recent global fit [51]. For $\sin^2\theta_{13}$, we take the $3\sigma$ range in the normal mass ordering.

TABLE 1: Best fit data of the parameters $\theta$, $\sin\theta_{ij}$, and $\delta_{\mathrm{CP}}$.

| Order | $\chi^2_{\min}$ | $\theta_{\mathrm{bf}}$ | $(\sin^2\theta_{13})_{\mathrm{bf}}$ | $(\sin^2\theta_{23})_{\mathrm{bf}}$ | $(\sin^2\theta_{12})_{\mathrm{bf}}$ | $(\delta_{\mathrm{CP}})_{\mathrm{bf}}$ |
|---|---|---|---|---|---|---|
| Normal | 4.856 | $\pm 0.131\pi$ | 0.0216 | 0.5 | 0.341 | $\mp\pi/2$ |
| Inverted | 5.855 | $\pm 0.132\pi$ | 0.0220 | 0.5 | 0.341 | $\mp\pi/2$ |

Correspondingly, for $X_{1\nu}$ we have

$$C^{+}_{\mathrm{Magic}} X_{1\nu} C_{\mathrm{Magic}} = X_{1\nu}, \; S^{+}_{23} X_{1\nu} S_{23} = X_{2\nu}. \tag{25}$$

$S_{23}$ works as the GCP for the mass matrix $M^{+}_{\nu} M_{\nu}$ on the one hand. On the other hand, it acts as an equivalent transformation for symmetries $X_{1\nu}$ and $X_{2\nu}$. So $X_{1\nu}$ is equivalent to the residual symmetry $Z_2^{\mathrm{Magic}} \times \mathrm{CP}$.

*3.3. Comments on Equivalence of Elements of $S_4$ Group Algebras and $Z_2 \times CP$.* For the $S_4$ group with the GCP, the residual symmetries $Z_2 \times \mathrm{CP}$ could bring maximal or trivial Dirac CP phase. We have seen that $X_\nu \cong Z_2 \times \mathrm{CP}$ in $S_3$ group algebras gives a maximal CP phase.

In fact, the equivalence can still hold for some $X$ in $S_4$ group algebras which are not elements of $S_3$ group algebras. The trivial CP phase could be obtained from $X$. Here, we give an example of $X$ from $S_4$ group algebras with a different representation. Three generators of $S_4$ which satisfy the relation [32]

$$\begin{aligned} S^2 = V^2 = (SV)^2 = (TV)^2 = I, \\ T^3 = (ST)^3 = I, \\ (STV)^4 = I, \end{aligned} \tag{26}$$

are expressed as [32]

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad V = \mp\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{27}$$

where $\omega = e^{i2\pi/3}$. A nontrivial example of the $S_4$ group algebra element could be $X = (TV)\cos\theta + i\sin\theta(STV)$. Its specific expression is of the form [46]

$$X(\theta) = \begin{pmatrix} \frac{i}{3}\sin\theta - \cos\theta & \frac{2}{3}e^{i\pi/6}\sin\theta & -\frac{2}{3}e^{-i\pi/6}\sin\theta \\ -\frac{2i}{3}\sin\theta & \frac{2}{3}e^{i\pi/6}\sin\theta & \frac{1}{3}e^{-i\pi/6}\sin\theta - e^{-i2\pi/3}\cos\theta \\ -\frac{2i}{3}\sin\theta & -\frac{1}{3}e^{i\pi/6}\sin\theta - e^{i2\pi/3}\cos\theta & -\frac{2}{3}e^{-i\pi/6}\sin\theta \end{pmatrix}. \tag{28}$$

If we take $X_\nu = X(\theta)$ and suppose that the mass matrix of charged leptons is diagonal, we can obtain the lepton mixing matrix written as

$$U = \mathrm{diag}\left(1, \omega, \omega^2\right) \cdot \begin{pmatrix} -\sqrt{\frac{2}{3}}c_1 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}}s_1 \\ \sqrt{\frac{1}{6}}c_1 + \sqrt{\frac{1}{2}}s_1 & \frac{1}{\sqrt{3}} & \sqrt{\frac{1}{6}}s_1 - \sqrt{\frac{1}{2}}c_1 \\ \sqrt{\frac{1}{6}}c_1 - \sqrt{\frac{1}{2}}s_1 & \frac{1}{\sqrt{3}} & \sqrt{\frac{1}{6}}s_1 + \sqrt{\frac{1}{2}}c_1 \end{pmatrix}, \tag{29}$$

where $c_1 \equiv \cos\theta_1$, $s_1 \equiv \sin\theta_1$, and $\theta_1$ is a parameter constrained by the mixing angle $\theta_{13}$. So the mixing pattern is of trimaximal form with a trivial Dirac CP-violating phase. For $X(\theta)$, we can verify that the following relation holds, i.e.,

$$C_1^+ X(\theta) C_1 = X(\theta), \tag{30}$$

where $C_1 = T^+ST$, $C_1^2 = I$, and $T^+C_1T = C_1^*$. So $C_1$ and $T$ are a $Z_2$ symmetry and the corresponding CP transformation, respectively. Following the methods used in GCP [30], the lepton mixing matrix from the residual symmetry $Z_2 \times \mathrm{CP}$ can be expressed as $U_a = \Omega R_{13}(\theta_1)P$, where $\Omega$ and $R_{13}(\theta_1)$ are expressed, respectively, as

$$\Omega = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{\omega}{\sqrt{6}} & \frac{\omega}{\sqrt{3}} & \frac{-\omega}{\sqrt{2}} \\ \frac{\omega^2}{\sqrt{6}} & \frac{\omega^2}{\sqrt{3}} & \frac{\omega^2}{\sqrt{2}} \end{pmatrix}, \quad R_{13}(\theta_1) = \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 \\ 0 & 1 & 0 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix}, \tag{31}$$

$P$ is a phase matrix which can be neglected in our case of Dirac neutrinos. In particular, the matrix $\Omega$ satisfies the relations as follows

$$\Omega^+ C_1 \Omega = \mathrm{diag}\,(-1, 1, -1), \quad T = \Omega\Omega^T. \tag{32}$$

We can check that the matrix $U_a$ from the $Z_2 \times \mathrm{CP}$ is just the $U$ shown in Equation (29). So $X(\theta)$ is equivalent to the symmetry $Z_2 \times \mathrm{CP}$ generated by $C_1$ and $T$. Furthermore, let us consider the element $X'(\theta) \equiv T^+X(\theta)T$. The lepton mixing matrix from $X'(\theta)$ is $U' = T^+U$. Since $T$ is a phase matrix, $U'$ is equivalent to $U$. So the CP transformation interchanges the equivalent elements $X(\theta)$ and $X'(\theta)$. Therefore, the observation from the case of the $S_3$ algebra still holds in this example of the $S_4$ group algebra.

*3.4. Discussion on General Residual Symmetries of the Charged Lepton Sector.* We have studied the case that the mass matrix $M_e^+M_e$ is diagonal. The corresponding symmetry of the charged lepton sector is $U(1) \times U(1) \times U(1)$, namely, $X_e = \mathrm{diag}\,(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3})$. Now we discuss a more general case that $X_e$ is expressed by an element of the $S_3$ group algebra. Because all the elements listed in Equation (14) give the same mixing matrix up to permutations of rows or columns, we can take $X_{1e} = S_{23}e^{i\theta_e S_{12}}$. Then, the matrix $U_e$ is of the form

$$U_e = \begin{pmatrix} \frac{e^{i\theta_{1e}} - c_e e^{i(\theta_e - \theta_{1e})}}{is_e\sqrt{N_{1e}}} & \frac{1}{\sqrt{3}} & \frac{e^{i\theta_{2e}} - c_e e^{i(\theta_e - \theta_{2e})}}{is_e\sqrt{N_{2e}}} \\ \frac{1}{\sqrt{N_{1e}}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{N_{2e}}} \\ \frac{e^{i(\theta_e - \theta_{1e})}}{\sqrt{N_{1e}}} & \frac{1}{\sqrt{3}} & \frac{e^{i(\theta_e - \theta_{2e})}}{\sqrt{N_{2e}}} \end{pmatrix}, \tag{33}$$

where $e^{i\theta_{1e}} \equiv \sqrt{1-s_e^2/4} - is_e/2$, $e^{i\theta_{2e}} \equiv -\sqrt{1-s_e^2/4} - is_e/2$, $s_e \equiv \sin\theta_e$, $c_e \equiv \cos\theta_e$, and $N_{je} \equiv 2 + (1 + c_e^2 - 2c_e \cos(\theta_e - 2\theta_{je}))/s_e^2$, $j = 1, 2$. With respect to the mixing matrix $U_{\mathrm{PMNS}} \equiv U_e^+ U_\nu$, we have an element $U_{\mathrm{PMNS}}(\alpha i) = 1$. Obviously, it does not satisfy the constraint of the global fit data of neutrino oscillations. So the combination of the residual symmetries $(X_{1e}, X_{1\nu})$ does not give a realistic lepton mixing patten in the case of $S_3$ group algebra. Furthermore, if $\theta_e$ is equal to 0, $X_{1e}$ is reduced to $S_{23}$. The corresponding matrix $U_e$ becomes

$$U_e' = \begin{pmatrix} 0 & -\sin\theta' & \cos\theta' \\ \frac{-1}{\sqrt{2}} & \frac{\cos\theta'}{\sqrt{2}} & \frac{\sin\theta'}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{\cos\theta'}{\sqrt{2}} & \frac{\sin\theta'}{\sqrt{2}} \end{pmatrix}, \quad (34)$$

where $\theta'$ is an angle variable from the degeneracy of the eigenvalues of $S_{23}$. Then, $U_{\mathrm{PMNS}}$ contains a zero element. This observation still holds when $S_{23}$ is replaced by $S_{12}$ or $S_{13}$. So the combination $(Z_{2e}, X_{1\nu})$ is not a viable choice for the residual symmetries of leptons. We can also check that $U_{\mathrm{PMNS}}$ from the combination $(Z_{3e}, X_{1\nu})$, where $Z_{3e}$ is generated by $S_{123}$ or $S_{132}$, does not satisfy the constraint of the global fit data of neutrino oscillations either. It contains an element which is equal to 1. Therefore, when the residual symmetry of the neutrino sector is $X_{1\nu}$ in the $S_3$ group algebra, we can only take $X_e = \mathrm{diag}\,(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3})$.

## 4. Conclusion

We have studied a new structure to describe partial symmetries of charged leptons and Dirac neutrinos. The residual symmetry is expressed by an element of group algebras. In our construction, a specific lepton mixing pattern corresponds to a set of equivalent residual symmetries which are expressed by elements of group algebras $X_i$. These equivalent symmetries $X_i$ can be interchanged through a transformation which corresponds to a residual CP symmetry. For $S_3$ group algebras and a special case of $S_4$ group algebras, we found that $X_i$ is equivalent to a residual symmetry $Z_2 \times \mathrm{CP}$. The corresponding lepton mixing matrix is trimaximal. It is a difficult mathematical problem for us to determine whether $X_i$ is equivalent to $Z_2 \times \mathrm{CP}$ in general cases. Even so, observations from simple examples could still give us some interesting clues: (a) The parameter in partial symmetries may be viewed as a quantity to measure how discrete symmetries are mixed in the residual symmetry. (b) A partial symmetry dependent on a continuous parameter may be equivalent to a discrete symmetry with GCP. (c) The elementary residual CP transformation could be a permutation matrix or a diagonal phase matrix. A general one may be a finite product of elementary ones. Therefore, despite stringent experiment data, we could still construct some novel partial symmetries to obtain viable lepton mixing patterns.

## Data Availability

The global fit data supporting this research paper are from previously reported studies, which have been cited. The processed data are freely available.

## Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

## Acknowledgments

This work is supported by the National Natural Science Foundation of China under grant nos. 11405101 and 11705113, the Guangxi Scientific Program Foundation under grant no. Guike AD19110045, and the Research Foundation of Guilin University of Technology under grant no. GUTQDJJ2018103.

## References

[1] Y. Fukuda, T. Hayakawa, E. Ichihara et al., "Evidence for oscillation of atmospheric neutrinos," *Physical Review Letters*, vol. 81, no. 8, pp. 1562–1567, 1998.

[2] Q. R. Ahmad, R. C. Allen, T. C. Andersen et al., "Direct evidence for neutrino flavor transformation from neutral-current interactions in the Sudbury Neutrino Observatory," *Physical Review Letters*, vol. 89, no. 1, article 011301, 2002.

[3] F. P. An, J. Z. Bai, A. B. Balantekin et al., "Observation of electron-antineutrino disappearance at Daya Bay," *Physical Review Letters*, vol. 108, no. 17, article 171803, 2012.

[4] T. Yanagida, "Horizontal symmetry and masses of neutrinos," *Progress in Theoretical Physics*, vol. 64, no. 3, article 1103, 1105 pages, 1980.

[5] K. S. Babu, E. Ma, and J. W. F. Valle, "Underlying $A_4$ symmetry for the neutrino mass matrix and the quark mixing matrix," *Physics Letters B*, vol. 552, no. 3-4, pp. 207–213, 2003.

[6] E. Ma, "$A_4$ symmetry and neutrinos with very different masses," *Physical Review D*, vol. 70, article 031901(R), 2004.

[7] X.-G. He, Y.-Y. Keum, and R. R. Volkas, "$A_4$ flavor symmetry breaking scheme for understanding quark and neutrino mixing angles," *Journal of High Energy Physics*, vol. 604, p. 39, 2006.

[8] C. S. Lam, "Mass independent textures and symmetry," *Physical Review D*, vol. 74, no. 11, article 113004, 2006.

[9] W. Grimus and L. Lavoura, "A model realizing the Harrison-Perkins-Scott lepton mixing matrix," *Journal of High Energy Physics*, vol. 1, p. 18, 2006.

[10] M.-C. Chen and S. F. King, "$A_4$ see-saw models and form dominance," *Journal of High Energy Physics*, vol. 6, p. 72, 2009.

[11] G. Altarelli, F. Feruglio, and L. Merlo, "Revisiting bimaximal neutrino mixing in a model with $S_4$ discrete symmetry," *Journal of High Energy Physics*, vol. 5, p. 20, 2009.

[12] G. Altarelli and F. Feruglio, "Discrete flavor symmetries and models of neutrino mixing," *Reviews of Modern Physics*, vol. 82, no. 3, pp. 2701–2729, 2010.

[13] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, and M. Tanimoto, "Non-abelian discrete symmetries in particle

physics," *Progress of Theoretical Physics Supplement*, vol. 183, pp. 1–163, 2010.

[14] S. Antusch and V. Maurer, "Large neutrino mixing angle $\theta_{13}^{\mathrm{MNS}}$ and quark-lepton mass ratios in unified flavor models," *Physical Review D*, vol. 84, no. 11, article 117301, 2011.

[15] R. de Adelhart Toorop, F. Feruglio, and C. Hagedorn, "Discrete flavour symmetries in light of T2K," *Physics Letters B*, vol. 703, no. 4, pp. 447–451, 2011.

[16] D. Hernandez and A. Y. Smirnov, "Lepton mixing and discrete symmetries," *Physical Review D*, vol. 86, no. 5, article 053014, 2012.

[17] G.-J. Ding, "TFH mixing patterns, large $\theta_{13}$ and $\Delta(96)$ flavor symmetry," *Nuclear Physics B*, vol. 862, no. 1, pp. 1–42, 2012.

[18] S. Antusch, "Models for neutrino masses and mixings," *Nuclear Physics B*, vol. 235-236, pp. 303–309, 2013.

[19] S. F. King and C. Luhn, "Neutrino mass and mixing with discrete symmetry," *Reports on Progress in Physics*, vol. 76, no. 5, article 056201, 2013.

[20] C. Hagedorn, A. Meroni, and L. Vitale, "Mixing patterns from the groups $\Sigma(\eta^{\varphi})$," *Journal of Physics A: Mathematical and Theoretical*, vol. 47, no. 5, article 055201, 2014.

[21] R. M. Fonseca and W. Grimus, "Classification of lepton mixing matrices from finite residual symmetries," *Journal of High Energy Physics*, vol. 2014, no. 9, p. 33, 2014.

[22] C.-Y. Yao and G.-J. Ding, "Lepton and quark mixing patterns from finite flavor symmetries," *Physical Review D*, vol. 92, no. 9, article 096010, 2015.

[23] D. A. Dicus, S.-F. Ge, and W. W. Repko, "Generalized hidden $Z_2$ symmetry of neutrino mixing," *Physical Review D*, vol. 83, no. 9, article 093007, 2011.

[24] S.-F. Ge, D. A. Dicus, and W. W. Repko, "$Z_2$ symmetry prediction for the leptonic Dirac CP phase," *Physics Letters B*, vol. 702, no. 4, pp. 220–223, 2011.

[25] S.-F. Ge, D. A. Dicus, and W. W. Repko, "Residual symmetries for neutrino mixing with a large $\theta_{13}$ and nearly maximal $\delta_{D}$," *Physical Review Letters*, vol. 108, no. 4, article 041801, 2012.

[26] P. F. Harrison and W. G. Scott, "Symmetries and generalisations of tri-bimaximal neutrino mixing," *Physics Letters B*, vol. 535, no. 1-4, pp. 163–169, 2002.

[27] W. Grimus and L. Lavoura, "A non-standard CP transformation leading to maximal atmospheric neutrino mixing," *Physics Letters B*, vol. 579, no. 1-2, pp. 113–122, 2004.

[28] G. J. Ding, S. F. King, and A. J. Stuart, "Generalised CP and $A_4$ family symmetry," *Journal of High Energy Physics*, vol. 2013, no. 12, p. 6, 2013.

[29] M. Holthausen, M. Lindner, and M. A. Schmidt, "CP and discrete flavour symmetries," *Journal of High Energy Physics*, vol. 2013, no. 4, p. 122, 2013.

[30] F. Feruglio, C. Hagedorn, and R. Ziegler, "Lepton mixing parameters from discrete and CP symmetries," *Journal of High Energy Physics*, vol. 2013, no. 7, p. 27, 2013.

[31] G. J. Ding, S. F. King, C. Luhn, and A. J. Stuart, "Spontaneous CP violation from vacuum alignment in $S_4$ models of leptons," *Journal of High Energy Physics*, vol. 2013, no. 5, p. 84, 2013.

[32] F. Feruglio, C. Hagedorn, and R. Ziegler, "A realistic pattern of lepton mixing and masses from S4 and CP," *The European Physical Journal C*, vol. 74, no. 2, article 2753, 2014.

[33] I. Girardi, A. Meroni, S. T. Petcov, and M. Spinrath, "Generalised geometrical CP violation in a $T'$ lepton flavour model," *Journal of High Energy Physics*, vol. 2014, no. 2, p. 50, 2014.

[34] C.-C. Li and G.-J. Ding, "Generalised CP and trimaximal $TM_1$ lepton mixing in $S_4$ family symmetry," *Nuclear Physics B*, vol. 881, pp. 206–232, 2014.

[35] S. F. King and T. Neder, "Lepton mixing predictions including Majorana phases from $\Delta(6n2)$ flavour symmetry and generalised CP," *Physics Letters B*, vol. 736, pp. 308–316, 2014.

[36] G.-J. Ding and Y.-L. Zhou, "Lepton mixing parameters from $\Delta(48)$ family symmetry and generalised CP," *Journal of High Energy Physics*, vol. 2014, no. 6, p. 23, 2014.

[37] C.-C. Li and G.-J. Ding, "Lepton mixing in $A_5$ family symmetry and generalized CP," *Journal of High Energy Physics*, vol. 2015, no. 5, p. 100, 2015.

[38] C. Hagedorn, A. Meroni, and E. Molinaro, "Lepton mixing from $\Delta(3n^2)$ and $\Delta(6n^2)$ and CP," *Nuclear Physics B*, vol. 891, pp. 499–557, 2015.

[39] A. Di Iura, C. Hagedorn, and D. Meloni, "Lepton mixing from the interplay of the alternating group $A_5$ and CP," *Journal of High Energy Physics*, vol. 2015, no. 8, p. 37, 2015.

[40] P. Ballett, S. Pascoli, and J. Turner, "Mixing angle and phase correlations from A5with generalized CP and their prospects for discovery," *Physical Review D*, vol. 92, no. 9, article 093008, 2015.

[41] J. Turner, "Predictions for leptonic mixing angle correlations and nontrivial Dirac *CP* violation from $A_5$ with generalized *CP* symmetry," *Physical Review D*, vol. 92, no. 11, article 116007, 2015.

[42] C.-C. Li and G.-J. Ding, "Deviation from bimaximal mixing and leptonic CP phases in $S_4$ family symmetry and generalized CP," *Journal of High Energy Physics*, vol. 2015, no. 8, p. 17, 2015.

[43] C.-C. Li, C.-Y. Yao, and G.-J. Ding, "Lepton mixing predictions from infinite group series $D_{9n,\ 3n}^{(1)}$ with generalized *CP*," *Journal of High Energy Physics*, vol. 2016, no. 5, p. 7, 2016.

[44] J.-N. Lu and G.-J. Ding, "Alternative schemes of predicting lepton mixing parameters from discrete flavor and *CP* symmetry," *Physical Review D*, vol. 95, no. 1, article 015012, 2017.

[45] S.-j. Rong, "Lepton mixing patterns from the group $\Sigma(36\times3)$ with a generalized *CP* transformation," *Physical Review D*, vol. 95, no. 7, article 076014, 2017.

[46] S.-j. Rong, "A novel mathematical construct for the family of leptonic mixing patterns," http://arxiv.org/abs/1703.09981.

[47] M. Barile, M. S. Moslehian, and E. W. Weisstein, *Group algebra*from MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/GroupAlgebra.html.

[48] P. F. Harrison and W. G. Scott, "$\mu$-$\tau$ reflection symmetry in lepton mixing and neutrino oscillations," *Physics Letters B*, vol. 547, no. 3-4, pp. 219–228, 2002.

[49] Z.-Z. Xing and Z.-H. Zhao, "A review of $\mu$-$\tau$ flavor symmetry in neutrino physics," *Reports on Progress in Physics*, vol. 79, no. 7, article 076201, 2016.

[50] W. Rodejohann and X.-J. Xu, "Trimaximal $\mu$–$\tau$ reflection symmetry," *Physical Review D*, vol. 96, no. 5, article 055039, 2017.

[51] P. F. de Salas, D. V. Forero, C. A. Ternes, M. Tortola, and J. W. F. Valle, "Status of neutrino oscillations 2018: $3\sigma$ hint for normal mass ordering and improved CP sensitivity," *Physics Letters B*, vol. 782, pp. 633–640, 2018.

[52] J. Beringer, J. . F. Arguin, R. M. Barnett et al., "Review of particle physics," *Physical Review D*, vol. 86, no. 1, article 010001, 2012.