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*Research Article*

# Inequalities for the Class of Warped Product Submanifold of Para-Cosymplectic Manifolds

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The aim of this paper is to study the warped product pointwise semislant submanifolds in the para-cosymplectic manifold with the semi-Riemannian metric. For which, firstly we provide the more generalized definition of pointwise slant submanifolds and related characterization results followed by the definition of pointwise slant distributions and pointwise semislant submanifolds. We also derive some results for different foliations on distribution, and lastly, we defined pointwise semislant warped product submanifold, given existence and nonexistence results, basic lemmas, theorems, and optimal inequalities for the ambient manifold.

## 1. Introduction

To generalize the Riemannian product manifolds, Bishop and O'Neil [1] introduced the concept of warped product for the manifolds with negative curvature and showed the surface of revolution as the simplest example of warped product manifold. The authors of [2–5] studied the warped product submanifolds for different manifolds. Warped product plays the beneficial role in encoding the universe, and the inequalities related to the second fundamental form with the warping function cover the wide as well as important section of it. These were firstly formulated by Chen in [6, 7]. Warped product for lightlike manifolds for the first time was studied in [8] and for semi-Riemannian manifold under the name PR-warped product on para-Kähler manifold in [9], where he derived the aforesaid inequalities for the case of semi-Riemannian metric. From there, the study on warped product escalates among geometers with also in view that the same has so many applications in the physics mainly in general relativity and black hole theory [10].

Beside this, the name slant submanifolds were introduced as the generalized version of holomorphic and totally real cases of submanifolds by Chen in [11]. Further, the theory extended to various manifolds with Riemannian as well as semi-Riemannian metric by many geometers. Later, in 2017, the authors in [12] defined the slant submanifolds irrespective of the writinger angle for the semi-Riemannian manifold and formulated three cases which are separately explained and achieved some effective results with bunch of examples. They defined it in terms of quotient $g(tX, tX)/g(JX, JX)$ which is constant for the case of slant submanifolds for every vector field $X$ (spacelike or timelike) on the submanifold $M$ of manifold $(\bar{M}, J, g)$. As slant and semislant submanifolds generalized to pointwise slant submanifolds (former called quasi slant) by Etayo in [13], Chen and Garay studied the same for the almost Hermitian case [14]. Sahin [15] defined pointwise semislant notion of submanifolds with an example. Recently, there are many interesting papers related with submanifold theory, singularity theory, classical differential geometry, etc. The readers can find more details about those techniques and theories in a series of papers [16–29]. Moreover, interdisciplinary research is one of the hottest trends in science; in the future work, we intend to apply and combine the techniques and results presented in [16–25] alongside with the methods in this paper to obtain more new results.

The paper is structured as follows: Section 2 contains the preliminary knowledge about ambient manifold, submanifold, and warped product with some important lemmas. Section 3 defines the pointwise slant submanifold, characterization lemma, and an example. Section 4 and Section 5 deal with the study of the pointwise slant distributions and pointwise semislant submanifolds, respectively. Section 6 includes the definition of warped product, some nonexistence results, lemmas, and theorems provided with an example. Finally, inequalities for the same submanifold are given in Section 7.

## 2. Preliminaries

*Definition 1.* A $(2m+1)$-dimensional smooth manifold $\widehat{M}$ admits $(\varphi, \xi, \eta, g)$ structure with $\varphi$ as a $(1,1)$-tensor field, $\xi$ as a characteristic vector field, $\eta$ as a globally differential 1-form, and $g$ as a semi-Riemannian metric named as an almost paracontact semi-Riemannian manifold $(\widehat{M}, \varphi, \xi, \eta, g)$ which satisfies

$$\varphi^2 = I - \eta \otimes \xi, \eta(\xi) = 1, \tag{1}$$

$$g(\cdot, \cdot) = -g(\varphi \cdot, \varphi \cdot) + \eta(\cdot)\eta(\cdot), \tag{2}$$

where $I$ represents an identity transformation of tangent space of $\widehat{M}$ and $\otimes$ represents a tensor product. A structure compatible semi-Riemannian metric "$g$" relates to $\eta$ as [30].

$$g(\cdot, \xi) = \eta(\cdot). \tag{3}$$

Equations (1) and (2) easily ensure the following:

$$\begin{aligned} \text{rank } (\varphi) &= 2\text{m}, \\ \varphi\xi &= 0, \\ \eta \circ \phi &= 0, \end{aligned} \tag{4}$$

$$g(\varphi \cdot, \cdot) + g(\cdot, \varphi \cdot) = 0. \tag{5}$$

Let $\Phi$ be the fundamental 2-form on $\widehat{M}$; then,

$$\Phi(\cdot, \cdot) = d\eta(\cdot, \cdot) = g(\cdot, \varphi \cdot). \tag{6}$$

Basis. An almost paracontact semi-Riemannian manifold always exists with a $\varphi$ − basis$\{E_i, E_i^{\mathring{a}}, \xi\}$, a certain type of local pseudoorthonormal basis which includes $E_i$, $\xi$ as space-like, and $E_i^{\mathring{a}} = \varphi E_i$ as timelike vector fields.

*Definition 2* (see [31]). An almost paracontact semi-Riemannian manifold $\widehat{M}$ is termed as para-cosymplectic if the forms $\eta$ and $\Phi$ are parallel with respect to the Levi-Civita connection $\widehat{\nabla}$ by

$$\begin{aligned} \widehat{\nabla}\eta &= 0, \\ \widehat{\nabla}\Phi &= 0. \end{aligned} \tag{7}$$

**Lemma 3.** *Let $\widehat{M}$ be a para-cosymplectic manifold with structure vector field $\xi \in \Gamma(T\widehat{M})$; then,*

$$\widehat{\nabla}_X \xi = 0, \tag{8}$$

$\forall X \in \Gamma(T\widehat{M})$.

*Proof.* Directly follow with the help of Equations (4) and (7) and covariant differentiation. □

*2.1. Submanifold.* Let $M$ be an isometrically immersed submanifold of a para-cosymplectic manifold $\widehat{M}$ with an induced nondegenerate metric $g$ (denoted metric by same symbol as on $\widehat{M}$), denoting $\nabla$ as Levi-Civita connection and $h$ as the second fundamental form on $M$. Thus, the Gauss-Weingarten formulas are

$$\widehat{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{9}$$

$$\widehat{\nabla}_X \zeta = -A_\zeta X + \nabla_X^\perp \zeta, \tag{10}$$

for $X, Y \in \Gamma(TM)$ (tangent bundle), and $\zeta \in \Gamma(TM^\perp)$ (normal bundle); $\nabla^\perp$ denotes normal connection, and $A_\zeta$ denotes shape operator associated with the normal section on $M$. The metric relation of $A_\zeta$ and $h$ is given as

$$g(A_\zeta X, Y) = g(h(X, Y), \zeta). \tag{11}$$

Every $X \in \Gamma(TM)$ is split as

$$\varphi X = tX + nX. \tag{12}$$

Similarly, every $\zeta \in \Gamma(TM^\perp)$ is split as

$$\varphi\zeta = t^\perp\zeta + n^\perp\zeta, \tag{13}$$

where $tX$ and $t^\perp\zeta$ ($nX$ and $n^\perp\zeta$) are the tangential parts (normal parts) of $\varphi X$ and $\varphi\zeta$, respectively. Based on Equation (12), the submanifold $M$ classifies as *anti-invariant* if $t = 0$ or *invariant* if $n = 0$ on $M$. After using Equation (12) in Equation (5), we get

$$g(X, tY) = -g(tX, Y). \tag{14}$$

Now, from Lemma 3 and Equation (11), we have our next result.

**Lemma 4.** *If $M$ is a submanifold immersed in a paracosymplectic manifold $\widehat{M}$ with structure vector field $\xi \in \Gamma(TM)$, then*

$$\begin{gathered}\nabla_X \xi = \nabla_\xi X = \nabla_\xi \xi = 0, \\ h(X, \xi) = 0, \\ A_\zeta \xi = 0, A_\zeta X \perp \xi,\end{gathered} \tag{15}$$

*for every $X \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^\perp)$.*

Next, let us take two semi-Riemannian manifolds $(M_B, g_1)$ and $(M_F, g_2)$ and a positive smooth function $f$ on $M_B$. Taking $M_B \times M_F$ as the product manifold along with canonical projections,

$$\begin{aligned}\sigma_1 &: M_B \times M_F \longrightarrow M_B, \\ \sigma_2 &: M_B \times M_F \longrightarrow M_F,\end{aligned} \tag{16}$$

such that $\sigma_1(p_B, p_F) = p_B$ and $\sigma_2(p_B, p_F) = p_F$ for any point $p = (p_B, p_F) \in M_B \times M_F$. Then, the product manifold $M_W = M_B \times_f M_F$ is called warped product if metric $g$ called the warped metric on $M_W$ can be formulated as

$$\begin{aligned}g(X, Y) = g_1(\sigma_1 * (X), \sigma_1 * (Y)) \\ + (f \circ \sigma_1)^2 g_2(\sigma_2 * (X), \sigma_2 * (Y)).\end{aligned} \tag{17}$$

For every $X, Y \in \Gamma(TM_W)$, "$*$" represents the derivation map, and we call $f$ as a *warping function*. Abstractly, the metric can be written as

$$g = g_1 + f^2 g_2, \tag{18}$$

where the warped product $M_W = M_B \times_f M_F$ is split into a product of the base space $M_B$ and the fiber space $M_F$, except that the fiber $M_F$ is warped [1, 32].

**Proposition 5** (see [32]). *The warped product submanifold $M_W = M_B \times_f M_F$ satisfies*

*(i) $\nabla_X Y = \Gamma(TM_B)$*

*(ii) $\nabla_X U = \nabla_U X = (Xf/f)U$*

*(iii) $\nabla_U V = \nabla'_V U = (g(U, V)/f)\nabla f$*

*for $X, Y \in \Gamma(TM_B)$ and $U, V \in \Gamma(TM_F)$, where $\nabla$ is the Levi-Civita connection on $M_W$, $\nabla'$ is the connection on $M_F$, and $\nabla f$ is the gradient of $f$ defined as $g(\nabla f, X) = Xf$.*

Further, let $\{e_1,\cdots,e_K, e_{K+1},\cdots,e_{2m+1}\}$ be a local orthonormal basis on $T\widehat{M}$ among which $\{e_1,\cdots,e_{K+1}\}$ are tangent to $M$ and $\{e_{K+2},\cdots,e_{2m+1}\}$ are normal to $M$. If we set

$$\begin{gathered}h^k_{xy} = g\left(h\left(e_x, e_y\right), e_k\right), \\ x, y \in \{e_1,\cdots,e_{K+1}\}, \\ k \in \{e_{K+2},\cdots,e_{2m+1}\},\end{gathered} \tag{19}$$

then we get

$$h\left(e_x, e_y\right) = \sum_{k=K+1}^{2m+1} \varepsilon_k h^k_{xy} e_r, \varepsilon_k = g\left(e_k, e_k\right), \tag{20}$$

where $h^k_{xy}$ are the coefficients of $h$. Accordingly, squared norm of the second fundamental form $h$ is defined as

$$\|h\|^2 = \sum_{x,y=1}^{K+1} \varepsilon_x \varepsilon_y g\left(h\left(e_x, e_y\right), h\left(e_x, e_y\right)\right). \tag{21}$$

## 3. Pointwise Slant Submanifolds

The semi-Riemannian manifold has difficulty of defining the Writinger angle as the vector fields may be timelike. Thus, the next definition is in the view of [12], generalizing the slant submanifold in our ambient semi-Riemannian manifold.

*Definition 6.* An isometrically immersed submanifold $M$ of an almost paracontact manifold $\widehat{M}$ is termed as pointwise slant if at every point $p \in M$, the quotient $g(tX, tX)/g(\varphi X, \varphi X) = \lambda(\theta)$ for $\theta \geq 0$ is independent of the choice of any nonzero spacelike or timelike vector $X \in M_p$, where $M_p = \{X \in T_p M : g(X, \xi) = 0\}$. For slant angle $\theta$, we say $\lambda(\theta)$ a slant coefficient.

*Remark 7.* The value of $\lambda(\theta)$ can be

(i) $\lambda(\theta) = \cosh^2(\theta) \in [1,\infty)$ for $|tX|/|\varphi X| > 1$; $tX$ is timelike or spacelike of each spacelike or timelike vector field $X$ adding $\theta > 0$

(ii) $\lambda(\theta) = \cos^2(\theta) \in [0, 1]$ for $|tX|/|\varphi X| < 1$; $tX$ is timelike or spacelike of each spacelike or timelike vector field $X$ adding $0 \leq \theta \leq 2\pi$

(iii) $\lambda(\theta) = -\sinh^2(\theta) \in (-\infty,0]$ for $tX$ is timelike or spacelike for any timelike or spacelike vector field $X$ adding $\theta > 0$

*Remark 8.* The special cases are as follows:

(i) The constant value of $\lambda(\theta)$ throughout $M$ implies $M$ is *slant submanifold* [11, 12]

(ii) The point $p \in M$ is called a *complex point* if $t \equiv \varphi$, which means that the slant coefficient $\lambda(\theta)$ is equal to 1. The submanifold with every point as complex point is *complex or holomorphic submanifold*

(iii) The point $p \in M$ is called a *totally real point* if $t \equiv 0$, which means that the slant coefficient $\lambda(\theta)$ is equal to 0. The submanifold with every point as totally real point is *totally real submanifold*

Furthermore, let us take the union of all $M_p$'s and denoting the same by

$$T^*M = \bigcup_{p \in M} \{X \in M_p | g(X, \xi) = 0\}. \tag{22}$$

**Lemma 9.** *The submanifold $M$ isometrically immersed in para-cosymplectic manifold $\widehat{M}$ is a pointwise slant submanifold if and only if on every point $p \in M$; there exists $\lambda \in (-\infty, \infty)$ for some $\theta \geq 0$ such that $t^2X = \lambda(\theta)X$ for each spacelike (or timelike) vector field $X \in M_p$.*

*Proof.* For each point $p$ of a pointwise slant submanifold $M$, the definition (22) follow as

$$g(tX, tX) = \lambda(\theta) g(\varphi X, \varphi X), \tag{23}$$

for $X \in T_pM$. With the use of Equations (5) and (14) and the condition that $X \in T_pM$ in Equation (23), we get the desired result. □

Proceeding further with some results which are not hard to prove, any pointwise slant submanifold $M$ satisfies

$$g(tX, tY) = \lambda(\theta) g(\varphi X, \varphi Y) = -\lambda(\theta) g(X, Y),$$
$$g(nX, nY) = (1 - \lambda(\theta)) g(\varphi X, \varphi Y) = -(1 - \lambda(\theta)) g(X, Y), \tag{24}$$

for $X, Y \in T^*M$.

**Proposition 10.** *The submanifold $M$ of a para-cosymplectic manifold $\widehat{M}$ is pointwise slant submanifold if and only if*

*(i) $t^\perp nX = (1 - \lambda(\theta))X$ and $n\,tX = -n^\perp nX$ for any spacelike (or timelike) vector field $X \in \Gamma(TM)$*

*(ii) $(n^\perp)^2 \zeta = \lambda(\theta)\zeta$ for nonlightlike normal vector field $\zeta$, where $\lambda(\theta)$ is the slant coefficient of $M$*

*Proof.* Assume $M$ as a pointwise slant submanifold.

(i) Then for every $X \in T^*M$, $\varphi^2 X = X$. On other way,

$$\varphi^2 X = t^2 X + n\,tX + t^\perp nX + n^\perp nX. \tag{25}$$

Equating tangential and normal parts and using Lemma 9, we can attain the result

(ii) Since $\zeta \in \Gamma(TM^\perp)$, thus there exists $X \in \Gamma(T^*M)$ as $M$ is pointwise slant submanifold such that $nX = \zeta$. Now, $(n')^2 \zeta = n^\perp n^\perp nX = -n^\perp n\,tX = n\,t^2 X = \lambda(\theta)\zeta$. The converse can be easily derived using same equations

□

**Theorem 11** (see [33]). *A totally geodesic and connected pointwise slant submanifold $M$ of a para-cosymplectic manifold $\widehat{M}$ is a slant submanifold.*

## 4. Pointwise Slant Distributions

Analogous to [34], we generalize slant distributions by defining pointwise slant distributions in $\widehat{M}$. Furthermore, we study some basic characterizations for the distributions on our ambient manifold.

*Definition 12.* A pointwise slant distribution $\mathfrak{D}$ on $M$ is a differentiable distribution for which the quotient $g(t_{\mathfrak{D}}X, t_{\mathfrak{D}}X)/g(\varphi X, \varphi X) = \lambda_{\mathfrak{D}}(\theta)$ is independent of the choice of any spacelike or timelike vector field $X \in \mathfrak{D}_p$. Here,

(i) $\mathfrak{D}_p$ is the distribution at point $p \in M$

(ii) $t_{\mathfrak{D}}X$ is the projection of $\varphi X$ on the distribution $\mathfrak{D}$

(iii) $\lambda_{\mathfrak{D}}(\theta)$ is the slant coefficient corresponding to the distribution $\mathfrak{D}$ on $M$ for $\theta \geq 0$, and the value of $\lambda_{\mathfrak{D}}(\theta)$ may be $\cos h^2\theta$, $\cos^2\theta$, or $\sin h^2\theta$

*Remark 13.*

(1) A pointwise slant distribution $\mathfrak{D}$ is *invariant* if $t_{\mathfrak{D}}X \equiv \varphi X$ with $\lambda_{\mathfrak{D}}(\theta) = 1$ or *anti-invariant* for $t_{\mathfrak{D}}X \equiv 0$ with $\lambda_{\mathfrak{D}}(\theta) = 0$. Other than these two cases, we call the distribution to be *proper pointwise slant distribution* [12]

(2) The distribution $\mathfrak{D}$ on $M$ is as follows [9, 31]:

(i) totally geodesic: if $h(X, Y) = 0$

(ii) involutive: if $[X, Y] \in \mathfrak{D}$

for every $X, Y \in \mathfrak{D}$.

**Corollary 14.** *The distribution $\mathfrak{D}$ on the submanifold $M$ is pointwise slant distribution if and only if there exists $\lambda_{\mathfrak{D}}(\theta)$ for $\theta \geq 0$ such that $(t_{\mathfrak{D}})^2 X = \lambda_{\mathfrak{D}}(\theta)X$ for any nonlightlike vector field $X \in \mathfrak{D}_p \subset T_pM$.*

*Proof.* The result follows similar to Lemma 9. □

## 5. Pointwise Semislant Submanifold

*Definition 15.* A submanifold $M$ of a para-cosymplectic $\widehat{M}$ is named as pointwise semislant submanifold if the set of complementary orthogonal distributions $\{\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_\lambda\}$ exists on $M$ and fulfills the listed conditions:

(i) $TM = \mathfrak{D}_{\mathfrak{T}} \oplus \mathfrak{D}_\lambda$

(ii) $\mathfrak{D}_{\mathfrak{T}}$ is $\varphi$-invariant distribution, i.e., $\mathfrak{D}_{\mathfrak{T}} \subseteq \mathfrak{D}_T$

(iii) $\mathfrak{D}_\lambda$ is a pointwise slant distribution having $\lambda(\theta)$ as a slant coefficient for $\theta \geq 0$

*Remark 16.* Further, submanifold $M$ is

(i) *proper pointwise semislant* when $\mathfrak{D}_{\mathfrak{T}} \neq 0, \mathfrak{D}_\lambda \neq \{0\}$ with nonconstant $\lambda(\theta)$

(ii) *proper slant submanifold* when $\mathfrak{D}_{\mathfrak{T}} = \{0\}$ and $\mathfrak{D}_\lambda \neq \{0\}$ with $\lambda(\theta)$ globally constant for $\theta$ [35]

(iii) *proper semi-invariant* when $\mathfrak{D}_{\mathfrak{T}} \neq \{0\}$ and $\mathfrak{D}_\lambda \neq \{0\}$ such that $tX \equiv 0$ for any $X \in \Gamma(\mathfrak{D}_\lambda)$ [12]

(iv) *invariant submanifold* when $\mathfrak{D}_\lambda = \{0\}$ [35]

(v) *anti-invariant submanifold* when $\mathfrak{D}_{\mathfrak{T}} = \{0\}$ and $tX \equiv 0$ for every $X \in \Gamma(\mathfrak{D}_\lambda)$ [35]

*Remark 17.* The decomposition of the tangent space can be expressed in two ways:

(i) If $\xi \in \Gamma(TM)$, the $TM = \langle \xi \rangle \oplus \mathfrak{D}_{\mathfrak{T}}{}' \oplus \mathfrak{D}_\lambda$

(ii) If $\xi \in \Gamma(TM^\perp)$, the $TM = \mathfrak{D}_{\mathfrak{T}}{}' \oplus \mathfrak{D}_\lambda$. Here, $\mathfrak{D}_{\mathfrak{T}}{}' = \{X \in \mathfrak{D}_{\mathfrak{T}} : g(X, \xi) = 0\} \subseteq \mathfrak{D}_{\mathfrak{T}}$. Thus, we have either $\mathfrak{D}_{\mathfrak{T}} = \mathfrak{D}_{\mathfrak{T}}{}'$ or $\mathfrak{D}_{\mathfrak{T}} = \langle \xi \rangle \oplus \mathfrak{D}_{\mathfrak{T}}{}'$ [4]

Denote $\mathscr{P}_T$ and $\mathscr{P}_\lambda$ as the projections, respectively, on the distributions $\mathfrak{D}_{\mathfrak{T}}$ and $\mathfrak{D}_\lambda$. Then, any $X \in \Gamma(TM)$ is split as

$$X = \mathscr{P}_T X + \mathscr{P}_\lambda X. \tag{26}$$

Operating $\varphi$, using Equation (12) and the case distribution $\mathfrak{D}_{\mathfrak{T}}$ which is $\varphi$-invariant on the previous equation, we concluded that

$$tX = t\mathscr{P}_T X + t\mathscr{P}_\lambda X \in \Gamma(TM), nX = n\mathscr{P}_\lambda X \in \Gamma\left(TM^\perp\right). \tag{27}$$

As $\mathfrak{D}_\lambda$ is pointwise slant distribution, by the consequences of Corollary 14, we obtain that

$$t^2 X = \lambda(\theta) X, \tag{28}$$

for $X \in \Gamma(\mathfrak{D}_\lambda)$ with $\lambda(\theta)$ as the slant coefficient. Clearly, for any point $p \in M$, if $\xi \in T_p M$, then

$$\varphi X = t\mathscr{P}_{\mathscr{T}}{}' X + t\mathscr{P}_\lambda X + n\mathscr{P}_\lambda X, \tag{29}$$

where $\mathscr{P}_{\mathscr{T}}{}'$ is the projection on the distribution $\mathfrak{D}_{\mathfrak{T}}{}'$. But this does not affect our result as $\xi$ disappears when $\varphi$ operates on $Z$.

However, the normal bundle denoted as $TM^\perp$ may be written as

$$TM^\perp = n\mathfrak{D}_\lambda \oplus \nu, \tag{30}$$

where $\nu$ represents the subspace of normal bundle that is invariant under $\varphi$.

**Lemma 18** (see [31]). *The shape operator $A$ of a proper pointwise semislant submanifold $M$ of para-cosymplectic manifold $\widehat{M}$ ensures the listed conditions:*

$$g\left(\varphi A_{\varphi U} S, X\right) = g(\nabla_S U, X), \tag{31}$$

$$A_{\varphi U} V = A_{\varphi V} U, \tag{32}$$

$$A_\zeta X = \varphi A_{\varphi\zeta} X, \tag{33}$$

$$g\left(A_\zeta X, U\right) = -g\left(A_{\varphi\zeta} \varphi X, U\right), \tag{34}$$

*for $S \in \Gamma(TM)$, $X \in \Gamma(T\mathfrak{D}_{\mathfrak{T}})$, $U, V \in \Gamma(T\mathfrak{D}_\lambda)$, and $\zeta \in \Gamma(TM^\perp)$.*

Both when $\xi$ is normal or tangent to $M$, the integrability and geodesic conditions brought out to be same after calculations for both the distributions, thus denoting them as common $\mathfrak{D}_{\mathfrak{T}}$.

**Lemma 19.** *If $M$ is a proper pointwise semislant submanifold of para-cosymplectic manifold $\widehat{M}$, for $\xi \in \Gamma(TM)$ or $\xi \in \Gamma(TM^\perp)$, the invariant distribution $\mathfrak{D}_{\mathfrak{T}}$ on $M$ is*

*(i) integrable if and only if $h(tX, Y) = h(X, tY)$*

*(ii) totally geodesic if and only if $A_{ntU} Y, = A_{nU} tY$*

*for $X, Y \in \Gamma(\mathfrak{D}_{\mathfrak{T}})$ and $U \in \Gamma(\mathfrak{D}_\lambda)$.*

*Proof.* Equation (2) expands as

$$g([X, Y], U) = -g\left(\varphi\left(\widehat{\nabla}_X Y - \widehat{\nabla}_Y X\right), \varphi U\right), \tag{35}$$

for every nonzero vector fields $X, Y \in \Gamma(\mathfrak{D}_{\mathfrak{T}})$ and $U \in \Gamma(\mathfrak{D}_\lambda)$. Using Equation (12) for the $\varphi U$ in Equation (35) and followed by using Equations (5), (7), and (9) and Lemma 9, we arrive at

$$(1 - \lambda(\theta)) g([X, Y], U) = g(h(X, tY) - h(Y, tX), nU). \tag{36}$$

Result (i) is clear using remark (28) as $\lambda(\theta)$ is nonconstant in Equation (36). Again, from Gauss formula and Equation (2),

$$g(\nabla_X Y, U) = g\left(\widehat{\nabla}_X Y, U\right) = -g\left(\varphi \widehat{\nabla}_X Y, \varphi U\right). \tag{37}$$

Employing Equations (7), (9), (11), (12), and (28) and Remark 16 in Equation (37), result (ii) follows. □

**Lemma 20.** *If $M$ is a proper pointwise semislant submanifold of para-cosymplectic manifold $\widehat{M}$, for $\xi \in \Gamma(TM)$ or $\xi \in \Gamma(TM^{\perp})$, the pointwise slant distribution $\mathfrak{D}_{\lambda}$ on $M$ is*

*(i) involutive if and only if*

$$g(A_{nV}U - A_{nU}V, tX) = g(A_{ntU}V - A_{ntV}U, X) \tag{38}$$

*(ii) totally geodesic if and only if*

$$g(A_{nV}tX, U) = g(A_{ntV}X, U) \tag{39}$$

*for $X \in \Gamma(\mathfrak{D}_{\mathfrak{T}})$ and $U, V \in \Gamma(\mathfrak{D}_{\lambda})$.*

*Proof.* Equation (2) implies

$$g([U, V], X) = -g(\varphi[U, V], \varphi X) + \eta([U, V])\eta(X), \tag{40}$$

for every nonzero vector fields $X \in \Gamma(\mathfrak{D}_{\mathfrak{T}})$ and $U, V \in \Gamma(\mathfrak{D}_{\lambda})$. Solving separately the term $\{g(\varphi[U, V], \varphi X)\}$ using Equations (5), (7), (10), (12), and (28), we receive

$$\begin{aligned} g(\varphi[U, V], \varphi X) = {} & \lambda(\theta) g([U, V], X) \\ & + g\left(\lambda'(\theta)V(\theta)U - \lambda'(\theta)U(\theta)V, X\right) \\ & - g(A_{nV}U - A_{nU}V, \varphi X) \\ & + g(A_{ntV}U - A_{ntU}V, X), \end{aligned} \tag{41}$$

where $\lambda'(\theta)$ is the first derivative of $\lambda(\theta)$. Surely, $U, V$ are orthogonal to $X$ after using this fact in Equation (41), and substituting in Equation (40), we get

$$\begin{aligned} (1 - \lambda(\theta)) g([U, V], X) = {} & g(-A_{ntV}U + A_{ntU}V, X) \\ & + g(A_{nV}U - A_{nU}V, tX) \\ & + \eta([U, V])\eta(X). \end{aligned} \tag{42}$$

For $\xi \in \Gamma(TM)$, one can replace $X$ by $\xi$ in Equation (42), and consequently, we get

$$-\lambda(\theta) g([U, V], \xi) = g(h(U, \xi), ntV) - g(h(V, \xi), ntU). \tag{43}$$

Using Lemma 4 in Equation (43) and for reason that $\lambda(\theta)$ a nonconstant, we get

$$\eta([U, V]) = 0. \tag{44}$$

Therefore, in Equation (42) using Equation (44) along with the facts that $M$ is proper, we arrived at the desired result (i).

Further, using Gauss formula and employing Equations (2), (7), (9), (11), (12), and (28) give

$$\begin{aligned} g(\nabla_U V, X) = {} & \lambda(\theta) g(\nabla_X Y, U) + g\left(\lambda'(\theta)U(\theta)V, X\right) \\ & + g(A_{ntV}X, U) - g(A_{nV}tX, U)\eta\left(\widehat{\nabla}_U V\right)\eta(X). \end{aligned} \tag{45}$$

Since $\xi \in \Gamma(TM)$, we can replace $X$ by $\xi$ in Equation (45), and consequently, we get

$$\begin{aligned} (1 - \lambda(\theta))\eta(\nabla_V U) &= -g(A_{ntU}\xi, V) + \eta(\nabla_V U), \\ (-\lambda(\theta))\eta(\nabla_V U) &= -g(A_{ntU}\xi, V). \end{aligned} \tag{46}$$

Using Lemma 4 in above expression, we get

$$\eta(\nabla_V U) = \eta\left(\widehat{\nabla}_V U\right) = 0. \tag{47}$$

Hence, Equation (45) implies that

$$(1 - \lambda(\theta))(\nabla_U V, X) = g(A_{ntV}X, U) - g(A_{nV}tX, U). \tag{48}$$

Thus, from (48) and $M$ as proper, $X, Y, U$ as nonnull vector fields, the proof of the (ii) directly follows. □

## 6. Pointwise Semislant Warped Product Submanifold

*Definition 21.* A pointwise semislant warped product submanifold $M$ of a para-cosymplectic manifold $\widehat{M}$ is a warped product of an invariant submanifold $M_{\mathfrak{T}}$ and a proper pointwise slant submanifold $M_{\lambda}$ either in the form $M_{\mathfrak{T}} \times_f M_{\lambda}$ or $M_{\lambda} \times_f M_{\mathfrak{T}}$, where $f$ is a positive smooth function taken on first submanifold in the product and slant coefficient of $M_{\lambda}$ is $\lambda(\theta)$. A trivial product is the case of such submanifold for which warping function $f$ is constant.

**Proposition 22** (see [33]). *A nontrivial pointwise semislant warped product submanifold $M$ of the form $M_{\lambda} \times_f M_{\mathfrak{T}}$ with $\xi \in \Gamma(TM^{\perp})$ does not exist on a para-cosymplectic manifold $\widehat{M}$.*

**Proposition 23** (see [33]). *A nontrivial pointwise semislant warped product submanifold $M$ of the form $M_{\lambda} \times_f M_{\mathfrak{T}}$ with $\xi \in \Gamma(TM)$ does not exist on a para-cosymplectic manifold $\widehat{M}$.*

**Proposition 24.** *A nontrivial pointwise semislant warped product submanifold $M$ of the form $M_{\mathfrak{T}} \times_f M_{\lambda}$ with $\xi \in \Gamma(TM_{\lambda})$ does not exist on a para-cosymplectic manifold $\widehat{M}$.*

*Proof.* Directly follow from Lemma 4 and Proposition 5. □

**Lemma 25.** *For a nontrivial pointwise semislant warped product submanifold $M = M_{\mathfrak{T}} \times_f M_\lambda$ of a para-cosymplectic manifold $\widehat{M}$,*

$$g(h(X, U), ntU) - g(h(X, tU), nU) = 0, \tag{49}$$

$$g(h(X, tU), nU) = \lambda(\theta)(X \ln f) g(U, U), \tag{50}$$

$\forall X \in \Gamma(D_{\mathfrak{T}})$ *and* $U \in \Gamma(D_\lambda)$.

*Proof.*

(1) Since $g(h(X, U), ntU) = g(\widehat{\nabla}_X U, ntU)$ for $X \in \Gamma(TM_{\mathfrak{T}})$ and $U \in \Gamma(M_\lambda)$, on right side, using Equations (5), (7), and (12), Proposition 5, and Lemma 9, we have

$$\begin{aligned} g(h(X, U), ntU) = &-(X \ln f) g(tU, tU) \\ &- g\left(\widehat{\nabla}_X nU, tU\right) \\ &- \lambda(\theta)(X \ln f) g(U, U). \end{aligned} \tag{51}$$

Equations (10), (11), and (14) and Lemma 9 further help to achieve (49)

(2) As $g(h(X, tU), nU) = g(\widehat{\nabla}_X tU, nU)$, next substituting $nU = \varphi U - tU$ applying Equations (5) and (7), Proposition 5, and Lemma 9 and the facts that $M_{\mathfrak{T}}$ is invariant, we get (50)

□

**Proposition 26.** *If $M = M_{\mathfrak{T}} \times_f M_\lambda$ is a nontrivial pointwise semislant warped product submanifold of a para-cosymplectic manifold $\widehat{M}$, then*

$$(1 - \lambda(\theta)) g\left(\widehat{\nabla}_X U, Y\right) = g(h(X, tY), nU) - g(h(X, Y), ntU), \tag{52}$$

$$(1 - \lambda(\theta)) g\left(\widehat{\nabla}_U V, X\right) = g(h(U, tX), nV) - g(h(U, X), ntV), \tag{53}$$

$\forall X, Y \in \Gamma(D_{\mathfrak{T}})$ *and* $U, V \in \Gamma(D_\lambda)$.

*Proof.* As $g(\widehat{\nabla}_X U, Y) = -g(\varphi \widehat{\nabla}_X U, \varphi Y), \eta(\widehat{\nabla}_X U) = -g(\widehat{\nabla}_X \xi, U) = 0$. Using Equations (7), (9), and (12) and Lemma 9 in above expression gives (52). In similar way, we can prove (53). □

**Lemma 27.** *A nontrivial proper pointwise semislant submanifold $M = M_{\mathfrak{T}} \times_f M_\lambda$ of a para-cosymplectic manifold $\widehat{M}$ satisfies*

$$g(h(X, Y), nU) = 0, \tag{54}$$

$$g(h(X, V), nU) = -\varphi X(lnf) g(V, U) - X(\ln f) g(tV, U), \tag{55}$$

$$\begin{aligned} g(h(X, tV), nU) = &-\varphi X(\ln f) g(tV, U) \\ &- \lambda(\theta) X(\ln f) g(V, U), \end{aligned} \tag{56}$$

$$\begin{aligned} g(h(X, V), ntU) = &-\varphi X(\ln f) g(V, tU) \\ &+ \lambda(\theta) X(\ln f) g(V, U), \end{aligned} \tag{57}$$

$\forall X, Y \in \Gamma(D_{\mathfrak{T}})$ *and* $U, V \in \Gamma(D_\lambda)$.

*Proof.* Result (54) is not hard to prove using Equations (7), (12), and (14) and Proposition 5. Substituting $tU = V$ in Equation (49) gives $g(h(X, V), nU) = g(h(X, U), nV)$; one can replace $nV = \varphi V - tV$, and using Equations (5) and (7) and Proposition 5 gives (55). Putting $V = tV$ and $U = tU$, respectively, in Equation (55) gives results (56) and (57). □

**Lemma 28.** *If $M = M_{\mathfrak{T}} \times_f M_\lambda$ is a nontrivial pointwise semislant warped product submanifold of a para-cosymplectic manifold $\widehat{M}$, then*

*(i) for* $\xi \in \Gamma(TM_{\mathfrak{T}})$,

$$\begin{aligned} g(h(tX, V), nU) = &-(X - \eta(X)\xi)(\ln f) g(V, U) \\ &- \varphi X(\ln f) g(tV, U) \end{aligned} \tag{58}$$

*(ii) for* $\xi \in \Gamma(TM^\perp)$,

$$\begin{aligned} g(h(tX, V), nU) = &-X(\ln f) g(V, U) \\ &- \varphi X(\ln f) g(tV, U) \end{aligned} \tag{59}$$

$\forall X \in \Gamma(D_{\mathfrak{T}})$ *and* $U, V \in \Gamma(D_\lambda)$.

*Proof.* Replacing $X = \varphi X$ in Equation (55) and having the fact that submanifold $M_{\mathfrak{T}}$ is invariant, both results directly follow. □

**Proposition 29.** *Let $M = M_{\mathfrak{T}} \times_f M_\lambda$ be nontrivial pointwise semislant warped product submanifold of a para-cosymplectic manifold $\widehat{M}$, then*

$$g(h(X, V), ntU) - g(h(X, tU), nV) = \lambda'(\theta) X(\theta) g(U, V), \tag{60}$$

$\forall X \in \Gamma(D_{\mathfrak{T}})$ *and* $U, V \in \Gamma(D_\lambda)$, *and $\lambda'(\theta)$ is the first derivative of slant coefficient.*

*Proof.* Using metric and para-cosymplectic condition

$$g\left(\widehat{\nabla}_X U, V\right) = -g\left(\widehat{\nabla}_X \varphi U, \varphi V\right), \tag{61}$$

this expression under the effect of Equations (5), (9), (12), and (14) turns as

$$g\left(\widehat{\nabla}_X U, V\right) = -g\left(\widehat{\nabla}_X tU, tV\right) - g(h(X, tU), nV) + g\left(\widehat{\nabla}_X t^{\perp} nU + n^{\perp} nU, V\right). \tag{62}$$

Further, using Propositions 10 and (5) ended with the desired result. □

**Proposition 30.** *A nontrivial pointwise semislant warped product submanifold $M = M_{\mathfrak{T}} \times_f M_\lambda$ of a para-cosymplectic manifold $\widehat{M}$ satisfies the following:*

*(i) For $\xi \in \Gamma(TM)$,*

$$g(h(tX, V), nU) + g(h(X, V), ntU) = -[(1 - \lambda(\theta))X - \eta(X)\xi](\ln f)g(V, U) \tag{63}$$

*(ii) For $\xi \in \Gamma(TM^{\perp})$,*

$$g(h(tX, V), nU) + g(h(X, V), ntU) = -[(1 - \lambda(\theta))X](\ln f)g(V, U) \tag{64}$$

*$\forall X \in \Gamma(TM_{\mathfrak{T}})$ and $U, V \in \Gamma(TM_\lambda)$.*

*Proof.* Lemma 28 and Equation (57) of Proposition 26 directly give the results. □

*Definition 31.* The submanifold $M = M_{\mathfrak{T}} \times_f M_\lambda$ is named as mixed totally geodesic if for every $X \in \Gamma(D_{\mathfrak{T}})$ and $U \in \Gamma(D_\lambda)$ ,

$$h(X, U) = 0. \tag{65}$$

**Theorem 32.** *If $M = M_{\mathfrak{T}} \times_f M_\lambda$ is a mixed totally geodesic pointwise semislant warped product submanifold of a para-cosymplectic manifold $\widehat{M}$, following cases arise:*

*(i) If $\xi \in \Gamma(TM)$, then $M$ is either a trivial product or a warped product of a holomorphic (complex) submanifold and a totally real submanifold*

*(ii) If $\xi \in \Gamma(TM^{\perp})$, then $M$ is either a trivial product or a warped product of two complex submanifolds*

*Proof.* Using definition (59), $M$ satisfies $h(X, U) = 0$ as well as $h(\varphi X, U) = 0$ (as $M_{\mathfrak{T}}$ is $\varphi$-invariant) for $X \in \Gamma(TM_{\mathfrak{T}})$ and $U \in \Gamma(TM_\lambda)$. Using this condition in proposition (58) when $\xi \in \Gamma(TM)$, we get

$$[(1 - \lambda(\theta))]X - \eta(X)\xi](\ln f)g(V, U) = 0. \tag{66}$$

Indicate either $\ln f = 0$ implies the trivial case or $[(1 - \lambda(\theta)]X = \eta(X)\xi$, after taking inner product with $\xi$, and in the view of Remark 8, the condition for the totally real holds for the submanifold $M_\lambda$. Following similar way for the second case, we ended up with $\lambda(\theta) = 1$ which is the condition for complex submanifold. □

**Theorem 33.** *A mixed totally geodesic pointwise semislant warped product submanifold $M = M_{\mathfrak{T}} \times_f M_\lambda$ of a para-cosymplectic manifold $\widehat{M}$ satisfies*

$$\varepsilon(ntU \ln f)\|X\|^2 = g\left(\left(\widehat{\nabla}_X t\right)X, tU\right), \tag{67}$$

*$\forall$ spacelike (or timelike) vector fields $X \in \Gamma(TM_{\mathfrak{T}})$ and $U \in \Gamma(TM_\lambda)$.*

*Proof.* As $g(\widehat{\nabla}_X tX, tU) = g(\widehat{\nabla}_X t)X, tU) + g(t\widehat{\nabla}_X X, tU)$, under the effects of Equation (14), Proposition 5, and Lemma 9, it turns

$$g\left(\widehat{\nabla}_X tX, tU\right) = g\left(\widehat{\nabla}_X t\right)X, tU\right) + \lambda(\theta)(U \ln f)g(X, X). \tag{68}$$

Other way, $g(\widehat{\nabla}_X tX, tU) + g(\widehat{\nabla}_X nX, tU) = g(\widehat{\nabla}_X \varphi X, tU)$; with this expression under the use of Equations (5), (7), and (12), Proposition 5, and Lemma 9, $g(\widehat{\nabla}_X nX, tU) = 0$ and $M$ as mixed totally geodesic, we have

$$g\left(\widehat{\nabla}_X tX, tU\right) = \lambda(\theta)(U \ln f)g(X, X) + \varepsilon(ntU \ln f)\|X\|^2. \tag{69}$$

This expression with the use of Equation (68) and for the reason vector field $X$ can be spacelike or timelike yields the result. □

*Example 34.* Consider a 7-dimensional smooth manifold $\widehat{M} = \mathbb{R}^6 \times \mathbb{R}^+ \subset \mathbb{R}^7$ having standard Cartesian coordinates as $(y_1, y_2, y_3, y_4, y_5, y_6, t)$ and defining a structure $(\varphi, \xi, \eta, g)$ as

$$\begin{aligned} &\varphi e_1 = e_4, \varphi e_2 = e_5, \varphi e_3 = e_6, \varphi e_4 = e_1, \varphi e_5 = e_2, \\ &\varphi e_6 = e_3, \varphi e_7 = 0, \\ &\xi = e_7, \eta = dt \text{ and } g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) \\ &= g(e_7, e_7) = 1, \\ &g(e_4, e_4) = g(e_5, e_5) = g(e_6, e_6) = -1, \end{aligned} \tag{70}$$

for $\{e_1, \cdots, e_7\}$ as a local orthonormal frame on $\Gamma(T\widehat{M})$. Obviously, $\widehat{M}$ over $(\varphi, \xi, \eta, g)$ fufills the condition of para-cosymplectic manifold. Let $M$ be a submanifold of $\widehat{M}$ with

$\xi$ tangent to it and defined for $x, y \in \mathbb{R}$, $u, v \geq 0$ and some $k$(constant) as

$$M(x, y, u, v, z) = ((x+y)\cosh u, (x+y)\sinh v, u, (x+y) \cdot \sinh u, (x+y)\cosh v, v, kx - ky + z). \quad (71)$$

The vector fields that generates the tangent bundle $TM$ are

$$\begin{aligned} X &= \cosh u e_1 + \sinh v e_2 + \sinh u e_4 + \cosh v e_5 + k e_7, \\ Y &= \cosh u e_1 + \sinh v e_2 + \sinh u e_4 + \cosh v e_5 - k e_7, \\ U &= (x+y)\sinh u e_1 + e_3 + (x+y)\cosh u e_4, \\ V &= (x+y)\cosh v e_2 + (x+y)\sinh v e_5 + e_6, \\ Z &= e_7. \end{aligned} \quad (72)$$

After calculations, it is found that the invariant distributions $(\mathfrak{D}_{\mathfrak{T}} \oplus \triangleleft\xi\triangleright)$ is the span of subspace $\{X, Y, Z\}$ and pointwise slant distribution $\mathfrak{D}_\lambda$ is the span of subspace $\{U, V\}$ with $t^2 = (1/(1 - x^2 - y^2)^2) Id$ such that $x^2 + y^2 \neq 1$; then, the slant coefficient is

(i) $\lambda(\theta) = \cosh^2\theta$ for $x^2 + y^2 < 1$

(ii) $\lambda(\theta) = \cos^2\theta$ for $x^2 + y^2 > 1$

As the distributions $(\mathfrak{D}_{\mathfrak{T}} \oplus \triangleleft\xi\triangleright)$ and $\mathfrak{D}_\lambda$ are integrable, let $M_T$ and $M_\lambda$ be their respective integral manifolds such that $M = M_{\mathfrak{T}} \times_f M_\lambda$ turns a nontrivial 5-dimensional pointwise semislant warped product submanifold of $\widehat{M}$ with induced metric $g$(semi-Riemannian) as

$$g = k^2 dX^2 + k^2 dY^2 + dZ^2 + (1 - x^2 - y^2)\{dU^2 - dV^2\}, \quad (73)$$

with warping function $f = \sqrt{(1 - x^2 - y^2)}$.

Let $M$ be an another submanifold of $\widehat{M}$ with $\xi$ normal to it and defined for $x, y, u, v \in \mathbb{R}$ as

$$M(x, y, u, v) = (xu, x, u + yv, yu, u + v, xv, 0). \quad (74)$$

Then, the vector fields that generates $TM$ are

$$\begin{aligned} X &= u e_1 + e_2 + v e_6, \\ Y &= v e_3 + u e_4 + e_5, \\ U &= x e_1 + e_3 + y e_4 + e_5, \\ V &= y e_3 + x e_6). \end{aligned} \quad (75)$$

The invariant distribution $\mathfrak{D}_{\mathfrak{T}}$ is the span of subspace $\{X, Y\}$, and pointwise slant distribution $\mathfrak{D}_\lambda$ is the span of subspace $\{U, V\}$ with $t^2 = (x^2/(x^2 - y^2)^2) I$ such that $x^2 \neq y^2$; then, the slant coefficient is

(i) $\lambda(\theta) = \cosh^2\theta$ for $x^2 > y^2$

(ii) $\lambda(\theta) = \cos^2\theta$ for $y^2 > x^2$

As the distributions $\mathfrak{D}_{\mathfrak{T}}$ and $\mathfrak{D}_\lambda$ are integrable, let $M_T$ and $M_\lambda$ be their respective integral manifolds such that $M = M_{\mathfrak{T}} \times_f M_\lambda$ turns a nontrivial 4-dimensional pointwise semislant warped product submanifold of $\widehat{M}$ with induced metric $g$(semi-Riemannian) as

$$g = (1 + u^2 - v^2) dX^2 - (1 + u^2 - v^2) dY^2 + (x^2 - y^2)\{dU^2 - dV^2\}, \quad (76)$$

with warping function $f = \sqrt{(x^2 - y^2)}$.

## 7. Inequalities

This section includes the geometric sharp inequalities for the aforesaid submanifold $M = M_{\mathfrak{T}} \times_f M_\lambda$ for the case $\xi$ tangent and normal to $M$.

**Lemma 35** (see [31]). *Let $M = M_{\mathfrak{T}} \times_f M_\lambda$ be a pointwise semislant warped product submanifold of a para-cosymplectic manifold $\widehat{M}$. Then, $M$ ensures*

$$g(h(X, U), \zeta) = -g(h(X, \varphi U), \varphi\zeta), \quad (77)$$

$$g(h(X, U), \varphi\zeta) = -g(\nabla_X^\perp \varphi U, \zeta), \quad (78)$$

$$g(h(X, U), \varphi\zeta) = -g(\nabla_Z^\perp \varphi U, \zeta), \quad (79)$$

*$\forall X \in \Gamma(\mathfrak{D}_{\mathfrak{T}})$, $U \in \Gamma(\mathfrak{D}_\lambda)$, and $\zeta \in \Gamma(\nu)$.*

**Theorem 36.** *Let $M = M_{\mathfrak{T}} \times_f M_\lambda$ be a pointwise semislant warped product submanifold of a para-cosymplectic manifold $\widehat{M}$ with $\xi \in \Gamma(TM)$. If $M_{\mathfrak{T}}$ is an invariant submanifold of $(2n_1 + 1)$-dimension and $M_\lambda$ is a proper pointwise slant submanifold $2n_2$-dimension satisfying $\nabla^\perp \varphi \mathfrak{D}_\lambda \subseteq \varphi \mathfrak{D}_\lambda$, the succeeding inequalities holds for $h$*

$$\begin{aligned} \|h\|^2 &\geq n_2 (1 + \lambda^2(\theta) \|\nabla \ln f\|^2 + \left\| h_\nu^{\mathfrak{D}_{\mathfrak{T}}} \right\|^2 \text{ for } S_1 \geq S_2, \\ \|h\|^2 &\leq n_2 (1 + \lambda^2(\theta) \|\nabla \ln f\|^2 + \left\| h_\nu^{\mathfrak{D}_{\mathfrak{T}}} \right\|^2 \text{ for } S_1 \leq S_2, \end{aligned} \quad (80)$$

*where $\lambda(\theta)$ is the slant coefficient corresponding to $M_\lambda$, $\nabla(\ln f)$ is the gradient of $\ln f$, $\|h_\nu^{\mathfrak{D}_{\mathfrak{T}}}\|^2 = g(h_\nu(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}}), h_\nu(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}}))$ with its $\nu$ component and invariant distribution $\mathfrak{D}_{\mathfrak{T}}$, $S_1 = (h_{rt}^s)^2 + (h_{r't'}^s)^2$, and $S_2 = (h_{r't}^s)^2 + (h_{rt'}^s)^2$.*

*Proof.* For $\xi \in \Gamma(TM)$, choose the local orthonormal frame on the following:

(a) $M_{\mathfrak{T}}$ by $\{e_i, e_{i'} = \varphi e_i\}$ for $i = \{1, \cdots, n_1$ and $e_i = \xi$ for $i = 2n_1 + 1$ such that $g(e_i, e_i) = \varepsilon_i = 1$ implies $g(e_{i'}, e_{i'}) = \varepsilon_{i'} = -1$ and $g(\xi, \xi) = \varepsilon_0 = 1$

(b) $M_\lambda$ by $\{\bar{e}_r, \bar{e}_{r'} = (1/\sqrt{|-\lambda(\theta)|})t\bar{e}_r\}$ for $r = \{1, \cdots, n_2\}$ and such that $g(\bar{e}_r, \bar{e}_r) = \bar{\varepsilon}_r = 1$ implies $g(\bar{e}_{r'}, \bar{e}_{r'}) = \bar{\varepsilon}_{r'} = -1$

(c) $(nM_\lambda)$ by $\bar{e}_s = (1/\sqrt{|-(1-\lambda(\theta))|})n\bar{e}_r$ for $r = \{1, \cdots, n_2\}$, having $g(\bar{e}_s, \bar{e}_s) = \bar{\varepsilon}_s = -1$ and on $\nu$ by $\{\zeta_l, \zeta_{l'} = \varphi\zeta\}$ such that $g(\zeta_l, \zeta_l) = \varepsilon_l = 1$ implies $g(\zeta_{l'}, \zeta_{l'}) = \varepsilon_{l'} = -1$

Compute $\|h\|^2$ which is given as

$$\|h\|^2 = \|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}})\|^2 + 2\|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_\lambda)\|^2 + \|h(\mathfrak{D}_\lambda, \mathfrak{D}_\lambda)\|^2. \tag{81}$$

The first term $\|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}})\|^2$ can be expanded as

$$\begin{aligned}\|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}})\|^2 &= g(h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}}), h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}})) \\ &= \sum_{i,j=0}^{n_1} \left[\varepsilon_i\varepsilon_j g\left(h\left(e_i, e_j\right), h\left(e_i, e_j\right)\right)\right. \\ &\quad + \varepsilon_{i'}\varepsilon_j g\left(h\left(e_{i'}, e_j\right), h\left(e_{i'}, e_j\right)\right) \\ &\quad + \varepsilon_i\varepsilon_{j'} g\left(h\left(e_i, e_{j'}\right), h\left(e_i, e_{j'}\right)\right) \\ &\quad \left. + \varepsilon_{i'}\varepsilon_{j'} g\left(h\left(e_{i'}, e_{j'}\right), h\left(e_{i'}, e_{j'}\right)\right)\right] \\ &\quad + \sum_{i=0}^{n_1} [\varepsilon_0\varepsilon_i g(h(e_0, e_i), h(e_0, e_i)) \\ &\quad + \varepsilon_i\varepsilon_0 g(h(e_i, e_0), h(e_i, e_o))].\end{aligned} \tag{82}$$

As $\mathfrak{D}_{\mathfrak{T}}$ is totally geodesic and Equation (54) of Lemma 27 directs that $h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}}) \in \nu$ using which, we can write

$$\begin{aligned}&h\left(e_i, e_j\right) = h_{ij}^l\zeta_l + h_{ij}^{l'}\zeta_{l'}, h\left(e_{i'}, e_j\right) = h_{i'j}^l\zeta_l + h_{i'j}^{l'}\zeta_{l'}, \\ &h\left(e_i, e_{j'}\right) = h_{ij'}^l\zeta_l + h_{ij'}^{l'}\zeta_{l'}, h\left(e_{i'}, e_{j'}\right) = h_{i'j'}^l\zeta_l + h_{i'j'}^{l'}\zeta_{l'}, \\ &h(e_0, e_i) = h_{0i}^l\zeta_l + h_{0i}^{l'}\zeta_{l'}, h(e_i, e_0) = h_{i0}^l\zeta_l + h_{i0}^{l'}\zeta_{l'}, \\ &h(e_0, e_0) = h_{00}^l\zeta_l + h_{00}^{l'}\zeta_{l'}.\end{aligned} \tag{83}$$

Simplifying these expressions in Equation (82) and using Equation (19) and Lemma 4 and in view of orthonormal frame, we get

$$\begin{aligned}\|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}})\|^2 = \sum_{i,j=1}^{n_1}\sum_{l=1}^{n_3}\Bigg[\left\{\left(h_{ij}^l\right)^2 - \left(h_{ij}^{l'}\right)^2\right\} - \left\{\left(h_{ij'}^l\right)^2 - \left(h_{i'j}^{l'}\right)^2\right\} \\ - \left\{\left(h_{ij'}^l\right)^2 - \left(h_{ij'}^{l'}\right)^2\right\} + \left(h_{i'j'}^l\right)^2 - \left(h_{i'j'}^{l'}\right)^2\Bigg\}\Bigg].\end{aligned} \tag{84}$$

The integrable condition of the $\mathfrak{D}_{\mathfrak{T}}$ and Equation (77) of the Lemma 35 implies that

$$\begin{aligned}&\left(h_{i'j}^l\right)^2 = \left(h_{ij'}^l\right)^2, \left(h_{i'j}^{l'}\right)^2 = \left(h_{ij'}^{l'}\right)^2, \\ &\left(h_{i'j'}^l\right)^2 = \left(h_{i'j'}^l\right)^2, \left(h_{i'j'}^{l'}\right)^2 = \left(h_{ij}^{l'}\right)^2,\end{aligned} \tag{85}$$

$$\left(h_{i'j}^l\right)^2 = \left(h_{i'j}^{l'}\right)^2, \left(h_{ij}^{l'}\right)^2 = \left(h_{i'j}^l\right)^2. \tag{86}$$

After substitution of Equation (84) in (86), we get

$$\|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\mathfrak{T}})\|^2 = 4\sum_{i,j=1}^{n_1}\sum_{l=1}^{n_2}\left[\left(h_{ij}^l\right)^2 - \left(h_{ij}^{l'}\right)^2\right] = \left\|h_\nu^{\mathfrak{D}_{\mathfrak{T}}}\right\|^2. \tag{87}$$

For the second part, we have

$$\begin{aligned}\|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_\lambda)\|^2 &= g(h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_\lambda), h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_\lambda)) \\ &= \sum_{i=1}^{n_1}\sum_{r=1}^{n_2}[\varepsilon_i\bar{\varepsilon}_r g(h(e_i, \bar{e}_r), h(e_i, \bar{e}_r)) \\ &\quad + \varepsilon_{i'}\bar{\varepsilon}_r g(h(e_{i'}, \bar{e}_r), h(e_{i'}, \bar{e}_r)) \\ &\quad + \varepsilon_i\bar{\varepsilon}_{r'} g(h(e_i, \bar{e}_{r'}), h(e_i, \bar{e}_{r'})) \\ &\quad + \varepsilon_{i'}\bar{\varepsilon}_{r'} g(h(e_{i'}, \bar{e}_{r'}), h(e_{i'}, \bar{e}_{r'})) \\ &\quad + \varepsilon_0\bar{\varepsilon}_r g(h(e_0, \bar{e}_r), h(e_i, \bar{e}_r)) \\ &\quad + \varepsilon_0\bar{\varepsilon}_{r'} g(h(e_0, \bar{e}_{r'}), h(e_0, \bar{e}_{r'}))],\end{aligned} \tag{88}$$

where

$$h(e_i, \bar{e}_r) = h_{ir}^s\bar{e}_s + h_{ir}^l\zeta_l + h_{ir}^{l'}\zeta_{l'}, h(e_{i'}, \bar{e}_r) = h_{i'r}^s\bar{e}_s + h_{i'r}^l\zeta_l + h_{i'r}^{l'}\zeta_{l'}, \tag{89}$$

$$\begin{aligned}h(e_i, \bar{e}_{r'}) &= h_{ir'}^s\bar{e}_s + h_{ir'}^l\zeta_l + h_{ir'}^{l'}\zeta_{l'}, h(e_{i'}, \bar{e}_{r'}) \\ &= h_{i'r'}^s\bar{e}_s + h_{i'r'}^l\zeta_l + h_{i'r'}^{l'}\zeta_{l'},\end{aligned} \tag{90}$$

$$\begin{aligned}h(e_0, \bar{e}_r) &= h_{0r}^s\bar{e}_s + h_{0r}^l\zeta_l + h_{0r}^{l'}\zeta_{l'}, h(e_0, \bar{e}_{r'}) \\ &= h_{0r'}^s\bar{e}_s + h_{0r'}^l\zeta_l + h_{0r'}^{l'}\zeta_{l'}.\end{aligned} \tag{91}$$

After simplifying Equation (88) using expressions in Equation (91), we get

$$\begin{aligned}\|h(\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_\lambda)\|^2 = \sum_{i=1}^{n_1}\sum_{r,s=1}^{n_2}\Big[(h_{ir}^s)^2 + (h_{i'r'}^s)^2 - (h_{ir'}^s)^2 - (h_{ir'}^s)^2 \\ + \left\{\left(h_{ir}^l\right)^2 + \left(h_{i'r'}^l\right)^2 - \left(h_{i'r}^l\right)^2 - \left(h_{ir'}^l\right)^2\right\} \\ - \left\{\left(h_{ir}^{l'}\right)^2 + \left(h_{i'r'}^{l'}\right)^2 - \left(h_{i'r}^{l'}\right)^2 - \left(h_{ir'}^{l'}\right)^2\right\}.\end{aligned} \tag{92}$$

Using Equations (55), (56), (58), (77), and (78), we have

$$\begin{aligned}
&h^s_{ir} = -e_{i'}(\ln f)g(\bar{e}_r, \bar{e}_s), h^s_{i'r} = -e_i(\ln f)g(\bar{e}_r, \bar{e}_s),\\
&h^s_{ir'} = -\lambda(\theta)e_i(\ln f)g(\bar{e}_r, \bar{e}_s),\\
&h^{l'}_{i'r'} = -\lambda(\theta)e_{i'}(\ln f)g(\bar{e}_r, \bar{e}_s), \left(h^l_{ir}\right)^2 = \left(h^{l'}_{ir'}\right)^2,\\
&\left(h^l_{i'r}\right)^2 = \left(h^{l'}_{i'r'}\right)^2 \text{ and } \left(h^{l'}_{ir}\right)^2 = \left(h^l_{ir'}\right)^2.
\end{aligned} \tag{93}$$

Substituting above values in Equation (92), we have

$$\begin{aligned}
\|h(\mathfrak{D}_\mathfrak{T}, \mathfrak{D}_\lambda)\|^2 &= n_2\left(1+\lambda^2(\theta)\right)\sum_{i=1}^{n_1}\left[(e_i(\ln f))^2 - (e_{i'}(\ln f))^2\right]\\
&\quad + 2\left[\left\{\left(h^l_{i'r}\right)^2 - \left(h^{l'}_{i'r}\right)^2\right\} - \left\{\left(h^l_{ir}\right)^2 - \left(h^{l'}_{ir}\right)^2\right\}\right].
\end{aligned} \tag{94}$$

Since $\sum_{i=1}^{n_1}[(e_i(\ln f))^2 - (e_{i'}(\ln f))^2] = g(\nabla(\ln f), \nabla(\ln f)) = \|\nabla(\ln f)\|^2$ and using the condition that $\nabla^\perp\varphi(D_\lambda) \subseteq \varphi(D_\lambda)$ in formula (78), we concluded that $h(\mathfrak{D}_\mathfrak{T}, \mathfrak{D}_\lambda) \subseteq \varphi(\mathfrak{D}_\lambda)$, above equation leads to

$$\|h(\mathfrak{D}_\mathfrak{T}, \mathfrak{D}_\lambda)\|^2 = n_2\left(1+\lambda^2(\theta)\right)\|\nabla(\ln f)\|^2. \tag{95}$$

Lastly,

$$\begin{aligned}
\left\|h\left(\mathfrak{D}_\lambda, \mathfrak{D}_\lambda\right)\right\|^2 &= g(h(\mathfrak{D}_\lambda, \mathfrak{D}_\lambda), h(\mathfrak{D}_\lambda, \mathfrak{D}_\lambda))\\
&= \sum_{r,t=1}^{n_2}[\bar{\varepsilon}_r\bar{\varepsilon}_t g(h(\bar{e}_r, \bar{e}_t), h(\bar{e}_r, \bar{e}_t))\\
&\quad + \bar{\varepsilon}_{r'}\bar{\varepsilon}_t g(h(\bar{e}_{r'}, \bar{e}_t), h(\bar{e}_{r'}, \bar{e}_t))\\
&\quad + \bar{\varepsilon}_r\bar{\varepsilon}_{t'} g(h(\bar{e}_r, \bar{e}_{t'}), h(\bar{e}_r, \bar{e}_{t'}))\\
&\quad + \bar{\varepsilon}_{r'}\bar{\varepsilon}_{t'} g(h(\bar{e}_{r'}, \bar{e}_{t'}), h(\bar{e}_{r'}, \bar{e}_{t'})),
\end{aligned} \tag{96}$$

where the included expressions are as below:

$$\begin{aligned}
h(\bar{e}_r, \bar{e}_t) &= h^s_{rt}\bar{e}_s + h^l_{rt}\zeta_l + h^{l'}_{rt}\zeta_{l'},\\
h(\bar{e}_{r'}, \bar{e}_t) &= h^s_{r't}\bar{e}_s + h^l_{r't}\zeta_l + h^{l'}_{r't}\zeta_{l'}\\
h(\bar{e}_r, \bar{e}_{t'}) &= h^s_{rt'}\bar{e}_s + h^l_{rt'}\zeta_l + h^{l'}_{rt'}\zeta_{l'},\\
h(\bar{e}_{r'}, \bar{e}_{t'}) &= h^s_{r't'}\bar{e}_s + h^l_{r't'}\zeta_l + h^{l'}_{r't'}\zeta_{l'}.
\end{aligned} \tag{97}$$

Employing these expressions in Equation (96) in view of the chosen frame and simplifying, we get

$$\begin{aligned}
\left\|h\left(\mathfrak{D}_\lambda, \mathfrak{D}_\lambda\right)\right\|^2 = \sum_{r,s,t=1}^{n_2}\Bigg[&\left\{\left(h^s_{rt}\right)^2 + \left(h^l_{rt}\right)^2 - \left(h^{l'}_{rt}\right)^2\right\}\\
&- \left\{\left(h^s_{r't}\right)^2 + \left(h^l_{r't}\right)^2 - \left(h^{l'}_{r't}\right)^2\right\}\\
&- \left\{\left(h^s_{rt'}\right)^2 + \left(h^l_{rt'}\right)^2 - \left(h^{l'}_{rt'}\right)^2\right\}\\
&+ \left\{\left(h^s_{r't'}\right)^2 + \left(h^l_{r't'}\right)^2 - \left(h^{l'}_{r't'}\right)^2\right\}\Bigg].
\end{aligned} \tag{98}$$

Using the condition that $\nabla^\perp\varphi(D_\lambda) \subseteq \varphi(D_\lambda)$ in formula (79), we concluded that $h(\mathfrak{D}_\lambda, \mathfrak{D}_\lambda) \subseteq \varphi(\mathfrak{D}_\lambda)$, which implies the Equation (98) with

$$\begin{aligned}
\left\|h\left(\mathfrak{D}_\lambda, \mathfrak{D}_\lambda\right)\right\|^2 = \sum_{r,s,t=1}^{n_2}\Big[&\left\{\left(h^s_{rt}\right)^2 + \left(h^s_{r't'}\right)^2\right\}\\
&- \left\{\left(h^s_{r't}\right)^2 + \left(h^s_{rt'}\right)^2\right\}\Big].
\end{aligned} \tag{99}$$

Result directly follows by letting $S_1 = (h^s_{rt})^2 + (h^s_{r't'})^2$ and $S_2 = (h^s_{r't})^2 + (h^s_{rt'})^2$. □

*Remark 37.* Equality holds if $S_1 = S_2$.

**Theorem 38.** *Let $M = M_\mathfrak{T} \times_f M_\lambda$ be a pointwise semislant warped product submanifold of a para-cosymplectic manifold $\widehat{M}$ with $\xi$ normal to $M$ such that $\xi \in \Gamma(\nu)$. If $M_\mathfrak{T}$ is an invariant submanifold of $2n_1$-dimension and $M_\lambda$ is a proper pointwise slant submanifold of $2n_2$-dimension satisfying $\nabla^\perp\varphi\mathfrak{D}_\lambda \subseteq \varphi\mathfrak{D}_\lambda$, the succeeding inequalities holds for $h$*

$$\begin{aligned}
\|h\|^2 &\geq n_2\left(1+\lambda^2(\theta)\|\nabla \ln f\|^2 + \left\|h_\nu^{\mathfrak{D}_\mathfrak{T}}\right\|^2 \text{ for } S_1 \geq S_2,\\
\|h\|^2 &\leq n_2\left(1+\lambda^2(\theta)\|\nabla \ln f\|^2 + \left\|h_\nu^{\mathfrak{D}_\mathfrak{T}}\right\|^2 \text{ for } S_1 \leq S_2,
\end{aligned} \tag{100}$$

*where $\lambda(\theta)$ is the slant coefficient corresponding to $M_\lambda$, $\nabla(\ln f)$ is the gradient of $\ln f$, $\|h_\nu^{\mathfrak{D}_\mathfrak{T}}\|^2 = g(h_\nu(\mathfrak{D}_\mathfrak{T}, \mathfrak{D}_\mathfrak{T}), h_\nu(\mathfrak{D}_\mathfrak{T}, \mathfrak{D}_\mathfrak{T}))$ with its $\nu$ component and invariant distribution $\mathfrak{D}_\mathfrak{T}$, $S_1 = (h^s_{rt})^2 + (h^s_{r't'})^2$, and $S_2 = (h^s_{r't})^2 + (h^s_{rt'})^2$.*

*Proof.* For $\xi \in \Gamma(TM^\perp)$, choose the local orthonormal frame on the following:

(a) $M_\mathfrak{T}$ by $\{e_i, e_{i'} = \varphi e_i\}$ for $i = \{1, \cdots, n_1\}$ such that $g(e_i, e_i) = \varepsilon_i = 1$ implies $g(e_{i'}, e_{i'}) = \varepsilon_{i'} = -1$

(b) $M_\lambda$ by $\{\bar{e}_r, \bar{e}_{r'} = (1/\sqrt{|-\lambda(\theta)|})t\bar{e}_r\}$ for $r = \{1, \cdots, n_2\}$ and such that $g(\bar{e}_r, \bar{e}_r) = \bar{\varepsilon}_r = 1$ implies $g(\bar{e}_{r'}, \bar{e}_{r'}) = \bar{\varepsilon}_{r'} = -1$

(c) $(nM_\lambda)$ by $\bar{e}_s = (1/\sqrt{|-(1-\lambda(\theta))|})n\bar{e}_r$ for $r = \{1, \cdots, n_2\}$, having $g(\bar{e}_s, \bar{e}_s) = \bar{\varepsilon}_s = -1$ and on $\nu$ by $\{\zeta_l, \zeta_{l'} = \varphi\zeta\}$ for $l = \{1, \cdots, n_3\}$ and $\zeta_l = \xi$ for $l = 2n_3 + 1$ such that $g(\zeta_l, \zeta_l) = \varepsilon_l = 1$ implies $g(\zeta_{l'}, \zeta_{l'}) = \varepsilon_{l'} = -1$ and $g(\xi, \xi) = \varepsilon_0 = 1$

Further, result can be acquired carrying the same steps as above proof and using Equations (33) and (34) of Lemma 18. □

## Data Availability

There is no data used for this manuscript.

## Conflicts of Interest

The authors declare no competing interest.

## Authors' Contributions

M.D. and S.K.S conceptualized the study. A.A. was responsible for the methodology. F.M. was responsible for the software. W.A.M.O, F.M., and M.D. were responsible for the validation. A.A. was responsible for the formal analysis. S.K.S. was responsible for the investigation. S.K.S was responsible for the resources. A.A. was responsible for the data curation. S.K.S and M.D wrote the original draft. A.A. and F.M. wrote, reviewed, and edited the manuscript. F.M. was responsible for the visualization. S.K.S. supervised the study. F.M. was responsible for the project administration. A.A was responsible for the funding acquisition. All authors have read and agreed to the published version of the manuscript.

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