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Research Article

Ground State Solution for a Fourth Order Elliptic Equation of Kirchhoff Type with Critical Growth in $\mathbb{R}^N$

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In this paper, we consider the following fourth order elliptic Kirchhoff-type equation involving the critical growth of the form $\begin{cases} \Delta^2 u - (a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u + V(x)u = (I_\alpha * F(u))f(u) + \lambda |u|^{2^{**}-2}u, \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases}$ where $a > 0$, $b \geq 0$, λ is a positive parameter, $\alpha \in (N-2, N)$, $5 \leq N \leq 8$, $V : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a potential function, and I_α is a Riesz potential of order α. Here, $2^{**} = 2N/(n-4)$ with $N \geq 5$ is the Sobolev critical exponent, and $\Delta^2 u = \Delta(\Delta u)$ is the biharmonic operator. Under certain assumptions on $V(x)$ and $f(u)$, we prove that the equation has ground state solutions by variational methods.

1. Introduction

In this article, we study the following fourth-order elliptic Kirchhoff-type equation involving the critical growth of the form:

$$\begin{cases} \Delta^2 u - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + V(x)u = (I_\alpha * F(u))f(u) + \lambda |u|^{2^{**}-2}u, \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{1}$$

where $a > 0$, $b \geq 0$, λ is a positive parameter, $5 \leq N \leq 8$, $V : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a potential function, and I_α is a Riesz potential of order $\alpha \in (N-2, N)$ defined by $I_\alpha = (\Gamma((N-\alpha)/2))/(\Gamma(\alpha/2)\pi^{N/2}2^\alpha |x|^{N-\alpha})$. Here, $2^{**} = 2N/(N-4)$ with $N \geq 5$ is the Sobolev critical exponent, and $\Delta^2 u = \Delta(\Delta u)$ is the biharmonic operator, that is, $\Delta^2 u = \sum_{i=1}^N (\partial^4/\partial x_i^4)u + \sum_{i\neq j}^N (\partial^4/\partial x_i^2 \partial x_j^2)u$. Besides, $V(x)$: $\mathbb{R}^N \longrightarrow \mathbb{R}$ is a potential function satisfying

(V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) \coloneqq V_0 > 0$

(V2) $\text{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < \infty$, where meas denotes the Lebesegue measure in $\mathbb{R}^N$ and V_0 and M are positive constants.

Furthermore, we suppose that the function $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies

$(f1)$ $f(t) = o(t^{\alpha/N})$ as $t \longrightarrow 0$

$(f2)$ $\lim_{|t|\longrightarrow+\infty} (f(t)/t^{(\alpha+2)/(N-2)}) = 0$

$(f3)$ $f(t)/t$ is increasing on $(0, +\infty)$ and decreasing on $(-\infty, 0)$

$(f4)$ $f(t)$ is increasing on $\mathbb{R}$

On the one hand, in 2012, by the variational methods, Wang and An [1] studied the following fourth-order equation of Kirchhoff type:

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x, u), & x \in \Omega, \\ u = \nabla u = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

and obtained the existence and multiplicity of solutions. Later, Cabada and Figueiredo [2] considered a class of generalized extensible beam equations with critical growth in $\mathbb{R}^N$ as follows:

$$\begin{cases} \Delta^2 u - M\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + u = \lambda f(u) + |u|^{2^{**}-2}u, x \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{3}$$

where $M : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous, $f \in C(\mathbb{R}, \mathbb{R})$, $N \geq 5$, and $\lambda > 0$ is a parameter. With the help of the minimax theorem and the truncation technique, the existence of nontrivial solutions of equation (3) is proved for λ sufficiently large. Recently, Song and Shi [3] proved the multiplicity of solutions for the following fourth-order elliptic equation with critical exponent

$$\begin{cases} \Delta^2 u - g\left(\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = \lambda h(x, u) + |u|^{2^{**}-2}u, & x \in \Omega, \\ u = \nabla u = 0, & x \in \partial\Omega, \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary, $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous, $h \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$, $N \geq 5$, and λ is a positive parameter. Soon after that, Liang et al. [4] obtained multiplicity of solutions to the following generalized extensible beam equation with critical growth:

$$\begin{aligned} &\Delta^2 u - M\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + V(x)u \\ &\quad = k(x)|u|^{q-2}u + \lambda|u|^{2^{**}-2}u, x \text{ in } \mathbb{R}^N. \end{aligned} \tag{5}$$

In fact, in the earlier time, Ma has already applied the variational methods to study the existence and multiplicity of solutions for the following fourth-order boundary value problem of Kirchhoff type:

$$u'''' - M\left(\int_0^1 |u'|^2 dx\right)u' = f(x, u), \quad x \in (0, L),$$

$$\begin{cases} u'''' - M\left(\int_0^1 |u'|^2 dx\right)u'' = q(x)f\left(x, u, u'\right)x \in (0, 1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{6}$$

For more details, readers can refer to [5, 6] and the references therein.

Actually, for the special case of problem (2) with $M(t) = a + bt$, then problem (2) is reduced to the following fourth-order elliptic equations of Kirchhoff type:

$$\begin{cases} \Delta^2 u - \left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = f(x, u), & x \in \Omega, \\ u = \nabla u = 0, & x \in \partial\Omega. \end{cases} \tag{7}$$

This problem is related to the stationary analog of the evolution equation of Kirchhoff type:

$$u_{tt} + \Delta^2 u - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u = f(x, t). \tag{8}$$

Dimensions one and two are relevant from the point of view of physics, engineering, and other sciences, because in those situations model (8) is considered a good approximation for describing nonlinear vibrations of beams or plates (see [7, 8]). Different approaches have been taken to attack this problem under various hypotheses on the nonlinearity. For example, very recently, Wang et al. [9] concentrated on the following Navier BVPs:

$$\begin{cases} \Delta^2 u - \lambda\left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = f(x, u), & x \in \Omega, \\ u = \nabla u = 0, & x \in \partial\Omega, \end{cases} \tag{9}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $\lambda, a, b > 0$. Applying mountain pass techniques and the truncation method, they obtained the existence of nontrivial solution to equation (9) for λ small enough when $f(x, u)$ satisfies some superlinear assumptions. For whole space $\mathbb{R}^N$, Song and Chen [10] studied the class of Schrödinger-Kirchhoff-type biharmonic problems:

$$\begin{cases} \Delta^2 u - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + V(x)u = f(x, u), x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{10}$$

where f satisfies the Ambrosetti-Rabinowitz type conditions. Under appropriate assumptions on V and f, the existence of infinitely many solutions is proved by using the symmetric mountain pass theorem.

On the other hand, in the past decades, many scholars have studied the following problem:

$$-\Delta u + V(x)u = \left(I_\alpha * |u|^p\right)|u|^{p-2}u, \tag{11}$$

which is called nonlinear Choquard type equation. For the physical background, we refer to [11–13] and the references therein. Mathematically, the existence and qualitative properties of solutions to equation (5) have been studied for decades by variational methods. See [11, 14, 15] for earlier results and [16–26] for a recent work.

Motivated by the works we mentioned above, especially by [4, 10], we consider the combination of equations (7) and (11) and extend to the general convolution case in $\mathbb{R}^N$. In our paper, we get the ground state solution of problem (1).

Our main results are as follows:

Theorem 1. *If V satisfies (V1)-(V2) and $f \in C^1(\mathbb{R}, \mathbb{R})$ verifies (f1)-(f4), then problem (1) has a ground state solution.*

For the convenience of expression, hereafter, we use the following notations:

$H := H^2(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N): |\nabla u|, \Delta u \in L^2(\mathbb{R}^N)\}$ *equipped with scalar product* $\langle u, v\rangle_H = \int_{\mathbb{R}^N}(\Delta u \Delta v + \nabla u \nabla v + uv)dx$

$E := \{u \in H : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty\}$ *equipped with scalar product* $\langle u, v\rangle = \int_{\mathbb{R}^N}(\Delta u \Delta v + \nabla u \nabla v + V(x)uv)dx$*, therefore,* $\|u\| = [\int_{\mathbb{R}^N}(|\Delta u|^2 + |\nabla u|^2 + V(x)u^2)dx]^{1/2}$

(i) *$L^s(\mathbb{R}^N)(1 \le s \le \infty)$ denotes the Lebesgue space with the norm $|u|_s = (\int_{\mathbb{R}^N}|u|^s dx)^{1/s}$*

(ii) *For any $u \in H^2(\mathbb{R}^N) \setminus \{0\}$, u_t is denoted as*

$$u_t = \begin{cases} 0, & t = 0, \\ \sqrt{t}u\left(\frac{x}{t}\right), & t > 0 \end{cases} \tag{12}$$

(iii) *$C, C_1, C_2, \cdots$ denote positive constants possibly different in different lines*

Remark 2. By the assumptions of V, it is obvious that the embedding $E^\circ H^2(\mathbb{R}^N)$ is continuous. Furthermore, the embedding $E^\circ L^r(\mathbb{R}^N)$ is continuous for $r \in [1, 2^{**})$, and compact for $r \in [1, 2^*)$, where $2^* = 2N/(N-2)$ if $N \ge 3$ and $2^* = \infty$ if $N = 1$ or 2. Thus, for each $s \in [2, 2^{**}]$, there is a $C_\varepsilon > 0$ such that $|u|_s \le C_\varepsilon \|u\|$ for all $u \in E$.

2. Preliminaries

In this section, we will give some very important lemmas.

Lemma 3. *Assume (f1)-(f4) hold, then we have the following:*

(1) *For all $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that $|f(t)| \le \varepsilon |t|^{\alpha/N} + C_\varepsilon |t|^{(\alpha+2)/(N-2)}$ and $|F(t)| \le \varepsilon |t|^{(N+\alpha)/N} + C_\varepsilon |t|^{(N+\alpha)/(N-2)}$*

(2) *For all $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that for every $p \in (2, 2^*)$, $|F(t)| \le \varepsilon(|t|^{(N+\alpha)/N} + |t|^{(N+\alpha)/(N-2)}) + C_\varepsilon |t|^{p(N+\alpha)/2N}$, and $|F(t)|^{2N/(N+\alpha)} \le \varepsilon(|t|^2 + |t|^{2N/(N-\alpha)}) + C_\varepsilon |t|^p$*

(3) *For any $s \ne 0$, $sf(s) > 2F(s)$ and $F(s) > 0$*

Proof. One can easily obtain the results by elementary calculation. □

Lemma 4 (Hardy-Littlewood-Sobolev inequality [27]). *Let $0 < \alpha < N$, $p, q > 1$ and $1 \le r < s < \infty$ be such that*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\alpha}{N}, \frac{1}{r} - \frac{1}{s} = \frac{\alpha}{N}. \tag{13}$$

(1) *For any $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$, one has*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\alpha}} dxdy\right| \le C(N, \alpha, p)\|f\|_{L^p(\mathbb{R}^N)}\|g\|_{L^q(\mathbb{R}^N)} \tag{14}$$

(2) *For any $f \in L^r(\mathbb{R}^N)$, one has*

$\|1/|\cdot|^{N-\alpha} * f\|_{L^s(\mathbb{R}^N)} \le C(N, \alpha, r)\|f\|_{L^r(\mathbb{R}^N)}$

Remark 5. By Lemma 3(1), Lemma 4(1) and Sobolev imbedding theorem, we can get

$$\begin{aligned} \left|\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)dx\right| &\le C|F(u)|^2_{2N/(N+\alpha)} \\ &\le C\left[\int_{\mathbb{R}^N}\left(|u|^{(N+\alpha)/N} + |u|^{(N+\alpha)/(N-2)}\right)^{2N/(N+\alpha)}dx\right]^{(N+\alpha)/N} \\ &\le C\left[\int_{\mathbb{R}^N}\left(|u|^2 + |u|^{2N/(N-2)}\right)dx\right]^{(N+\alpha)/N} \\ &\le C\left(\|u\|^{(2N+2\alpha)/N} + \|u\|^{(2N+2\alpha)/(N-2)}\right). \end{aligned} \tag{15}$$

3. Variational Formulation

The associated energy function of problem (1) is given by

$$\begin{aligned} I(u) = {} & \frac{1}{2}\int_{\mathbb{R}^N}|\Delta u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N}\left[a|\nabla u|^2 + V(x)u^2\right]dx \\ & + \frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2 dx\right)^2 - \frac{1}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)dx \\ & - \frac{\lambda}{2^{**}}\int_{\mathbb{R}^N}|u|^{2^{**}}dx \end{aligned} \tag{16}$$

i.e. the critical points of the functional $I(u)$ are weak solutions of problem (1). Under the assumptions, $I \in C^1(E, \mathbb{R})$, and for all $u, v \in E$, it holds that

$$\left\langle I'(u), v \right\rangle = \int_{\mathbb{R}^N} \Delta u \Delta v dx + a\int_{\mathbb{R}^N} \nabla u \nabla v dx + \int_{\mathbb{R}^N} V(x) uv dx + b\int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} \nabla u \nabla v dx - \int_{\mathbb{R}^N} (I_\alpha * F(u)) f(u) v dx - \lambda \int_{\mathbb{R}^N} |u|^{2^{**}-2} uv dx. \quad (17)$$

Thus,

$$\left\langle I'(u), u \right\rangle = \int_{\mathbb{R}^N} |\Delta u|^2 dx + \int_{\mathbb{R}^N} \left[a|\nabla u|^2 + V(x)u^2\right] dx + b\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^N} (I_\alpha * F(u)) f(u) u dx - \lambda \int_{\mathbb{R}^N} |u|^{2^{**}} dx. \quad (18)$$

In this section, we prove the following results.

Lemma 6. *The functional $I(u)$ possesses the mountain-pass geometry, i.e.,*

(1) There exists $\rho, \delta > 0$ such that $I(u) \ge \delta$ for all $\|u\| = \rho$

(2) There exists $e \in H^2(\mathbb{R}^N)$ such that $\|e\| > \rho$ and $I(e) < 0$

Proof. (1) By Lemma 3(1) and Lemma 4, we have

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \left[a|\nabla u|^2 + V(x)u^2\right] dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) dx - \frac{\lambda}{2^{**}} \int_{\mathbb{R}^N} |u|^{2^{**}} dx \ge C_1\|u\|^2 - C_2\left(\|u\|^{(2N+2\alpha)/N} + \|u\|^{(2N+2\alpha)/(N-2)}\right) - C_3\|u\|^{2^{**}}. \quad (19)$$

Thus, there exists $\rho, \delta > 0$ such that $I_\infty \ge \delta$ for all $\|u\| = \rho > 0$ small enough.

(2) For any $u \in E \setminus \{0\}$,

$$I(u_t) = \frac{t^{N-3}}{2}\int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{at^{N-1}}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{t^{N+1}}{2}\int_{\mathbb{R}^N} V(x)u^2 dx + \frac{bt^{2N-2}}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 - \frac{t^{N+\alpha}}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) dx - \frac{\lambda}{2^{**}} t^{\left(N^2+3N\right)/(N-4)} \int_{\mathbb{R}^N} |u|^{2^{**}} dx \longrightarrow -\infty, \quad (20)$$

as $t \longrightarrow \infty$, since $\alpha > N-2$; thus, we see $I_\infty(u_t) < 0$ for $t > 0$ large. Note that

$$\|u_t\|^2 = t^{N-3}\int_{\mathbb{R}^N} |\Delta u|^2 dx + t^{N-1}\int_{\mathbb{R}^N} |\nabla u|^2 dx + t^{N+1}\int_{\mathbb{R}^N} V(x)u^2 dx. \quad (21)$$

Taking $e = u_{t_0}$, with $t_0 > 0$ large, we have $\|e\| > \rho$ and $I_\infty(e) < 0$. □

Hence, we define the mountain-pass level of I:

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0, \quad (22)$$

where $\Gamma = \{\gamma \in C([0,1], E): \gamma(0) = 0, I(\gamma(1)) < 0\}$.

Lemma 7. *If $\{u_n\} \subset E$ is a $(PS)_c$ sequence of I, then $\{u_n\}$ is bounded in E.*

Proof. Let $\{u_n\} \subset E$ be $(PS)_c$ sequence, i.e., $I(u_n) \longrightarrow c$ and $I'(u_n) \longrightarrow 0$; then, we have

$$c = I(u_n) - \frac{1}{4}\left\langle I'(u_n), u_n \right\rangle + o(1) = \frac{1}{4}\int_{\mathbb{R}^N} |\Delta u_n|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N} \left[a|\nabla u_n|^2 + V(x)u_n^2\right] dx + \frac{1}{4}\int_{\mathbb{R}^N} (I_\alpha * F(u_n))[f(u_n)u_n - 2F(u_n)] dx + \left(\frac{\lambda}{4} - \frac{\lambda}{2^{**}}\right)\int_{\mathbb{R}^N} |u_n|^{2^{**}} dx + o(1) \ge C_4\|u_n\|^2. \quad (23)$$

Consequently, $\{u_n\}$ is bounded in E. □

Remark 8. By Lemma 7, we can assume that there exists a u such that

$$\begin{cases} u_n u \text{ in } E, \\ u_n \longrightarrow u \text{ in } L^s_{loc}\left(\mathbb{R}^N\right), \quad \forall s \in [2, 2^{**}), \\ u_n \longrightarrow u \; a.e. \text{on } \mathbb{R}^N. \end{cases} \quad (24)$$

Then, by the similar method as Lemma 3.3 in [4], we can obtain $\lim_{n \longrightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{2^{**}} dx = \int_{\mathbb{R}^N} |u|^{2^{**}} dx$.

Lemma 9. *$I(u)$ satisfies (PS) condition.*

Proof. Let $\{u_n\} \subset E$ be $(PS)_c$ sequence, i.e., $I(u_n) \longrightarrow c$ and $I'(u_n) \longrightarrow 0$; by Lemma 7, $\{u_n\}$ is bounded in E. Hence, up to a subsequence, we may assume that there exists a u such that

$$\begin{cases} u_n u \text{ in } E, \\ u_n \longrightarrow u \text{ in } L^s_{loc}(\mathbb{R}^N), \quad \forall s \in [2,2^{**}), \\ u_n \longrightarrow u \; a.e. \text{on } \mathbb{R}^N. \end{cases} \tag{25}$$

Then, using the lower semicontinuity of the norm, Brezis-Lieb Lemma [28] and Remark 8, we have

$$\begin{aligned} o(1)\|u_n\| = \left\langle I'(u_n), u_n \right\rangle &= \int_{\mathbb{R}^N} |\Delta u_n|^2 dx \\ &\quad + \int_{\mathbb{R}^N} \left[a|\nabla u_n|^2 + V(x)u_n^2\right] dx + b\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)^2 \\ &\quad - \int_{\mathbb{R}^N} (I_\alpha * F(u_n))f(u_n)u_n dx - \lambda \int_{\mathbb{R}^N} |u_n|^{2^{**}} dx \\ &= \int_{\mathbb{R}^N} |\Delta u_n - \Delta u|^2 dx + \int_{\mathbb{R}^N} \left[a|\nabla(u_n - u)|^2 \right. \\ &\quad \left. + V(x)|u_n - u|^2\right] dx + \int_{\mathbb{R}^N} |\Delta u|^2 dx + \int_{\mathbb{R}^N} \left[a|\nabla u|^2 \right. \\ &\quad \left. + V(x)u^2\right] dx + b\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)^2 \\ &\quad - \int_{\mathbb{R}^N} (I_\alpha * F(u_n))f(u_n)u_n dx \\ &\quad - \lambda \int_{\mathbb{R}^N} |u_n|^{2^{**}} dx + o(1) \\ &\geq C_5\|u_n - u\|^2 + \int_{\mathbb{R}^N} |\Delta u|^2 dx + \int_{\mathbb{R}^N} \left[a|\nabla u|^2 \right. \\ &\quad \left. + V(x)u^2\right] dx + b\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 \\ &\quad - \int_{\mathbb{R}^N} (I_\alpha * F(u))f(u)u dx - \lambda \int_{\mathbb{R}^N} |u|^{2^{**}} dx + o(1). \end{aligned} \tag{26}$$

Thus, we can have $\|u_n - u\|^2 \longrightarrow 0$, which implies that $\{u_n\}$ strongly converges to u in E. This completes the proof of Lemma 9. □

Remark 10. Now, we recall the Nehari manifold

$$\mathcal{M} := \left\{u \in E \setminus \{0\}: \left\langle I'(u), u \right\rangle = 0\right\}. \tag{27}$$

Let $m = \inf_{u \in \mathcal{M}} I(u)$; then for any $u \in \mathcal{M}$, we have

$$I(u) = I(u) - \frac{1}{4}\left\langle I'(u), u \right\rangle \geq C_6\|u_n\|^2 \geq 0. \tag{28}$$

Hence, m is well defined. Moreover, by the similar argument as Chapter 4 [29], we have the following characterization:

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) = m = \inf_{u \in \mathcal{M}} I(u). \tag{29}$$

4. Ground State Solution for Problem (1)

In this section, we prove the main theorem.

Proof of Theorem 11. From Lemmas 6 and 7, we know that there exists a bounded $(PS)_c$ sequence $\{u_n\}$, that is, $I(u_n) \longrightarrow c = m$, $I'(u_n) \longrightarrow 0$. Next, let $\delta := \limsup_{n\longrightarrow\infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_n|^2 dx$. We claim $\delta > 0$. On the contrary, by similar argument as Lions' concentration compactness principle, we can proof $u_n \longrightarrow 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^{**}$. By Lemma 3(2), for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\begin{aligned} &\limsup_{n\longrightarrow\infty} \int_{\mathbb{R}^N} (I_\alpha * F(u_n))f(u_n)u_n dx \\ &\quad \leq C \limsup_{n\longrightarrow\infty} \left[\varepsilon\left(\int_{\mathbb{R}^N} |u_n|^2 dx \right.\right. \\ &\qquad \left.\left. + \int_{\mathbb{R}^N} |u_n|^{2N/(N-2)} dx\right) + C_\varepsilon \int_{\mathbb{R}^N} |u_n|^p dx\right]^{(N+\alpha)/N} \\ &\quad \leq C\left[\varepsilon C_7 + C_\varepsilon \limsup_{n\longrightarrow\infty} \int_{\mathbb{R}^N} |u_n|^p dx\right]^{(N+\alpha)/N} \\ &\quad = C(\varepsilon C_8)^{(N+\alpha)/N}. \end{aligned} \tag{30}$$

Note that ε is arbitrary; we get

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n))f(u_n)u_n dx = o(1). \tag{31}$$

Combining with $I'(u_n) \longrightarrow 0$ and Remark 8, we can get

$$\begin{aligned} o(1) = \left\langle I'(u_n), u_n \right\rangle &= \int_{\mathbb{R}^N} |\Delta u_n|^2 dx + a\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \\ &\quad + b\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)^2 + \int_{\mathbb{R}^N} V(x)|u_n|^2 dx \\ &\quad - \int_{\mathbb{R}^N} (I_\alpha * F(u_n))f(u_n)u_n dx - \lambda \int_{\mathbb{R}^N} |u_n|^{2^{**}} dx \\ &\geq C_9\|u_n\|^2 - \int_{\mathbb{R}^N} (I_\alpha * F(u_n))f(u_n)u_n dx \\ &\quad - \lambda \int_{\mathbb{R}^N} |u_n|^{2^{**}} dx, \end{aligned} \tag{32}$$

which implies that

$$C_9\|u_n\|^2 \leq \int_{\mathbb{R}^N} (I_\alpha * F(u_n))f(u_n)u_n dx + \lambda \int_{\mathbb{R}^N} |u_n|^{2^{**}} dx + o(1). \tag{33}$$

Then, we have $\|u_n\| \longrightarrow 0$, which implies $u_n \longrightarrow 0$ in E. We deduce that $c = 0$, which contradicts to the fact that $c > 0$. Hence, $\delta > 0$, and there exists $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_n)} |u_n|^p dx \geq (\delta/2) > 0$. We set $v_n(x) = u_n(x + y_n)$; then, $\|u_n\| =$

$\|v_n\|, \int_{B_1(0)}|v_n|^p dx > \delta/2$ and $I(v_n) \longrightarrow c = m, I'(v_n) \longrightarrow 0$. Thus, there exists a $v_0 \neq 0$ such that

$$\begin{cases} v_n v_0 \text{ in } E, \\ v_n \longrightarrow v_0 \text{ in } L^s_{loc}\left(\mathbb{R}^N\right), \quad \forall s \in [2, 2^{**}) \\ v_n \longrightarrow v_0 \; a.e. \text{on } \mathbb{R}^N. \end{cases} \tag{34}$$

Then, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have $0 = \langle I'(v_n), \varphi \rangle + o(1) = \langle I'(v_0), \varphi \rangle$, which means v_0 is a solution of equation (1).

On the other hand, combining with the Fatou Lemma, we can obtain

$$\begin{aligned} m &= I(v_n) - \frac{1}{4}\left\langle I'(v_n), v_n \right\rangle + o(1) \\ &= \frac{1}{4}\int_{\mathbb{R}^N}|\Delta v_n|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N}\left[a|\nabla v_n|^2 + V(x)v_n^2\right]dx \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^N}(I_\alpha * F(v_n))[f(v_n)v_n - 2F(v_n)]dx \\ &\quad + \left(\frac{\lambda}{4} - \frac{\lambda}{2^{**}}\right)\int_{\mathbb{R}^N}|v_n|^{2^{**}}dx + o(1) \\ &\geq \frac{1}{4}\int_{\mathbb{R}^N}|\Delta v_0|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N}\left[a|\nabla v_0|^2 + V(x)v_0^2\right]dx \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^N}(I_\alpha * F(v_0))[f(v_0)v_0 - 2F(v_0)]dx \\ &\quad + \left(\frac{\lambda}{4} - \frac{\lambda}{2^{**}}\right)\int_{\mathbb{R}^N}|v_0|^{2^{**}}dx + o(1) \\ &= I(v_0) - \frac{1}{4}\left\langle I'(v_0), v_0 \right\rangle + o(1) = I(v_0) + o(1). \end{aligned} \tag{35}$$

At the same time, we know $m = c \leq I(v_0)$ by the definition of m. Then, we can deduce that v_0 is a ground state solution of equation (1). Thus, we complete the proof of Theorem 11. □

Data Availability

Data is not available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

LZ curated the data and is responsible for the writing of the original draft. CZ acquired funding and is assigned to the validation. LZ and CZ did the writing, review, and editing.

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