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Some Critical Notes on the Cantor Diagonal Argument

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Abstract

This paper critically examines the Cantor Diagonal Argument (CDA) that is used in set theory to draw a distinction between the cardinality of the natural numbers and that of the real numbers. The CDA is discussed here using a consensus from the forms found in a range of recently published sources. Four points critical of the CDA are raised that cast doubts on its validity and general applicability. Also, contrary to the conclusion conventionally drawn from the CDA, it is found possible to set up a one-to-one correspondence between the natural numbers and the real numbers. Finally, some comments are made on the concept of "infinity", pointing out that to consider this as an entity is a category error, since it simply represents an absence, that is, the absence of a termination to a process.

Keywords

Cantor Diagonal Argument, Infinity, Natural Numbers, One-to-One Correspondence, Real Numbers

1. Introduction

1) The concept of infinity is evidently of fundamental importance in number theory, but it is one that at the same time has many contentious and paradoxical aspects. The current position depends heavily on the theory of infinite sets and the concept of one-to-one correspondence that was introduced over a century ago by the German mathematician Georg Cantor (Anon, 2022; Cantor, 1874; Cantor, 1891; Cantor, 1915). In essence, the argument claims to show that a "new" real number can be produced which differ from those in a list of the real numbers when indexed against the natural numbers, so that this former list is not exhaustive.

2) Fundamentally, any discussion of this topic ought to start from a consideration of the work of Cantor himself, and in particular, his 1891 paper (Cantor,

1891) that is presumably to be considered the starting point for the CDA.

3) In fact, with this paper (Cantor, 1891), the relevant text on page 76 in the reference, shows that he considered here specifically an infinite set with two types of elements (m and w) in a specific order. This is unlike the present format of the CDA as conventionally presented and as discussed below, since no numbers are involved.

4) The discussion of the CDA used here therefore follows that of more recent published treatments that differ only in detail (Anon, 2022; Dantzig, 1954; Devlin, 1990; Elwes, 2010; Enderton, 1977; Gamow, 1988; Hemmings & Tahta, 1992; Hodges, 1998; Hofstadter, 1979; Hogben, 1960; Klein, 1932; Midonick, 1988; Northrop, 1947; Penrose, 1989; Young & Young, 1906). Following these, the real numbers are considered as their decimal expansion in the open interval (0,1).

5) In addition to these papers, three from the more recent literature, Kotani (2016), Livadas (2020) and Sharma (2021), are included in the References (although not discussed here specifically) to indicate that this is an area still subject to debate and critical comment.

6) It is useful, as points of comparison for the later discussion, to examine the ways in which infinite processes are normally treated, considering two elementary examples: the one-to-one mapping between the whole set of the natural numbers and the set of the even natural numbers; and the summation of a converging infinite series.

7) In the case of one-to-one mapping between the set of all natural numbers $\mathbb{N}$ and the set of the even natural numbers $\mathbb{N}'$, we have a sequence of the form shown in Scheme 1.

$\mathbb{N}$		$\mathbb{N}'$
1	$\Leftrightarrow$	2
2	$\Leftrightarrow$	4
3	$\Leftrightarrow$	6
4	$\Leftrightarrow$	8
	etc.	

Scheme 1.

At each stage, we have a positive outcome, in the sense that the pairing produced supports the contention of one-to-one mapping of the two series, and there is nothing to let us believe that this conclusion would alter if the process were extended indefinitely. Indeed, this may be treated without any consideration of infinity, since in the above listing each of the entries has the form: $n \Leftrightarrow 2n$, so that the one-to-one correspondence follows naturally.

8) As the second example, consider the process of the summation of a converging infinite series, for example:

$$S = 1/2 + 1/4 + 1/8 + \cdots$$

where the limit of the series is of course unity; here, as the summation is continued, the difference of the sum from unity exactly may be made as small as required by having enough terms, and this may be satisfied however small this difference is specified to be (the "epsilon-delta criterion"). Here again, a doubting questioner may be satisfied in a suitable way eventually, while the validity can be demonstrated by the standard mathematical proof ($2S = 1 + 1/2 + \cdots = 1 + S$, etc.). In any case, there is no reason to doubt that there would be any sudden change in this conclusion were the process continued indefinitely.

2. Basis of the CDA

1) To consider the basis of the CDA, the sources already cited (¶1.4) show somewhat diverse formats. In particular, some formats use symbols, and others use actual digits, in the expansion of the real numbers. It is simplest to consider one particular format, and then look at the differences from other formats that have been used. In fact, the format used here does not follow exactly that of the published examples, but is typical in most senses.

2) The format to be used is then that of the set of natural numbers $\mathbb{N}$ paired in succession with the set of real numbers $\mathbb{R}$ in the open interval (0,1), it is assumed that this process does eventually exhaust all the natural numbers.

3) The only requirement is that the real numbers in this interval may be presented by infinite decimals, each of the digits which is between 0 and 9 (that is, in this case, we are using base 10).

4) A typical example for the first three stages of this would be as in **Scheme 2**, in which certain. Numerals in the expansions of $\mathbb{R}$ are in bold type for the purposes of the CDA.

$\mathbb{N}$	$\mathbb{R}$
1	$0.\mathbf{1}23\cdots$
2	$0.4\mathbf{5}6\cdots$
3	$0.78\mathbf{9}\cdots$
etc.	etc.

Scheme 2.

5) The CDA then proceeds to construct a putatively new number by taking the diagonal formed by the digits in bold and then altering each of them in some specified way, such as by adding 1 in a cyclic fashion (0, 1, ⋯, 9, 0). This then produces a number designated here as X:

$$? \quad 0.260\cdots = X$$

where the indexing number shown as "?" is seemingly different from all those listed. This process may seemingly be continued indefinitely, and since the new number is different from all of those listed, then seemingly it is a "new" real number that has no matching natural number.

6) Conventionally, the conclusion that is then drawn is that there are more numbers in the list of real numbers $\mathbb{R}$ than there are in the list of natural numbers $\mathbb{N}$.

7) In the alternative case of the use of symbols, the sequence is interrupted at the general natural number n, as shown in Scheme 3 (where the distinction between the "etc1." and "etc2." is discussed below in ¶4.2) and the new number is formulated as

$$? \quad 0.b_1b_2b_3\cdots b_m\cdots b_n = B$$

where the Diagonal Argument is ensured by making

$$b_n \neq a_{mm}$$

with the same conclusion being drawn conventionally as in ¶2.6.

$\mathbb{N}$	$\mathbb{R}$
1	$0.\ a_{11}\ a_{12}\ a_{13}\ \cdots a_{1n}\ \cdots$
2	$0.\ a_{21}\ a_{22}\ a_{23}\ \cdots a_{2n}\ \cdots$
3	$0.\ a_{31}\ a_{32}\ a_{33}\ \cdots a_{3n}\ \cdots$
	etc1.
n	$0.\ a_{n1}\ a_{n2}\ a_{n3}\ \cdots a_{nn}\ \cdots$
	etc2.

Scheme 3.

3. Preliminary Comments on the CDA Formats

1) It is useful to make some preliminary comments on the CDA formats used in the published literature.

2) Considering the definition used in the formulation the real numbers in the interval (0.1), as already noted in any ¶2.3, any listing with the denary system (base ten) implicitly defines the sequence of entries after the decimal point as:

First entry:	Digit between 0 and 9
Second entry:	Digit between 0 and 9
Third entry:	Digit between 0 and 9
etc.	etc.

3) Considering the format used for the real numbers, in most cases of the methods published where digits rather than symbols are used, the real numbers are presented in the form: 0.123⋯; however, this is itself an infinity of numbers, since the termination "⋯" may have 0 to 9 as the first digit, 0 to 9 as the second digit, and so on. However, in the present case, to keep to the consensus of the published sources, it is necessary to accept any ambiguity in this termination.

4) The choice of the sequence of the real numbers in the range 0 to 1 in the published methods is often apparently arbitrary, and such a listing is therefore not evidently exhaustive. This applies whether the listing is given as actual digits, or as symbols. Following on the previous comments, it might therefore be preferable to list the real numbers (0,1) as mirroring the respective real number, which would ensure the real number list is complete, as shown in Scheme 4, where numbers highlighted in bold again form the diagonal, as discussed below. However, since this apparent lack of completeness is not thought a defect even in the "professional" presentations (Elwes, 2010; Enderton, 1977; Klein, 1932; Midonick, 1988), it will also be passed over here.

$\mathbb{N}$	$\mathbb{R}$
1	0.**1**000···
2	0.2**0**00···
3	0.30**0**0···
etc.	etc.

Scheme 4.

5) Where digits (rather than symbols) are used in the published examples, the numbers of places of decimals quoted vary widely, from three (Enderton, 1977), five (Hofstadter, 1979), eight (Gamow, 1988), ten (Penrose, 1989), to eleven (Anon, 2022) (where this last also uses binary rather than denary notation). It may be noted that one professional text (Enderton, 1977) takes the CDA to be so obvious that he does not restrict himself to the interval (0,1) but instead chooses the rather wildly chosen numbers: 236.001···, −7.777···, and 3.1415···.

6) These differences in the number of listed numbers also appear where symbols are used (Dantzig, 1954; Elwes, 2010; Klein, 1932) although here this may be rendered irrelevant by the use of the general entry for natural number n as in ¶2.7 above.

4. Two Critiques of the CDA

1) Two specific critiques are given here, which are independent in their approach, but concur in their results. Two critiques from other viewpoints are also given in §5 and §6 below.

2) In the first critique, we examine the conclusion that is drawn from the production of the putative new real number. It is stated that since this is different from any number on the list of real numbers, then this list is not exhaustive. However, there is a caveat that is omitted here, which is that *this only applies to numbers in the finite list as written*. For there is an indefinitely large number of entries below those specified, as indicated rather casually by the "etc." in ¶2.4 (**Scheme 2**) for examples where digits are used, and by the "etc2." in ¶2.7 (**Scheme 3**) where symbols are used. These represent real numbers still to be examined, to select a different entry at the specific place in the decimal expansion. This means that, however this far this process may be continued, at each stage there always remains an indefinitely large number of entries not yet considered.

3) Thus, the only proper conclusion to be drawn this stage, is that it cannot be excluded that the "diagonal number" X is on the remainder of the list, and that that applies however far the specified section of the list is extended.

4) By contrast, it seems that in the conventional approach, the construction of the "diagonal number" is viewed as a completed process, that is, the end is attained finally, even although this is accepted to be an infinite process that can never be completed. Moreover, the conclusions as given here in ¶4.3, that the "diagonal number" could be on the remainder of the list continue to apply at each stage in this process, but this seems to be abrogated at the assumed end of

the process.

5) This abrogation is in contrast to parallel "infinite" processes in mathematics, as in the correspondence between the set of all the natural numbers and the set of the even natural numbers (¶1.7), or the summation of a convergent series (¶1.8), where in each case there is no abrogation of the conclusions drawn were the processes to be carried without limit.

6) For the second critique, consider the derived number X in the light of the definition of the real numbers given above in ¶2.3 and again in ¶3.2. For its format shows that it fulfils the criterion that each entry in the number is a digit between 0 and 9. This in turn shows that this fulfils the definition of a real number between 0 and 1, and hence one that must be on the list, albeit lower down. Indeed, the initial premise that this is a list of the real numbers implies that it is a *complete* list (where this added adjective is really superfluous) so that the production of a "new" member represents a contradiction.

7) These critiques support one another, in that the uncertainty of the conclusion from the first critique, which indeed allows the "diagonal number" X to be further down the list, is resolved by the second critique, which shows that it must be present further down the list. Taken together, these critiques therefore cast doubt on the strength of the Cantor Diagonal Argument.

5. Exhaustive Application of the CDA

1) In its conventional formulation, the CDA is applied only once to obtain one "diagonal number" X. However, this restriction is arbitrary, and it is fruitful to examine the outcome of an exhaustive application of the CDA to the present example, allowing any change in the digits on the "diagonal" (¶5.2), and allowing any order for entries in the list (¶5.3). This would correspond to the results from one worker repeating the procedure in a random manner at intervals, or from numerous workers each taking their individual approach to the procedure.

2) The conventional approach, of course, leads to only one putatively "new" number, replacing one digit in each entry by a different one as specified. However, for completeness, consider all the cases where each digit in the "diagonal" is changed to one of the nine other possible digits. For the "diagonal" list of three entries in ¶2.4, there are $9 \times 9 \times 9 = 729$ distinct ways of changing the digits in the "diagonal", so that the exhaustive application of the CDA leads to 729 "new" numbers; in the general case, with n entries, this would be 9^n. Whatever the case, a list of "new" numbers is produced which is greater than that in the original list of real numbers according to a power factor. Furthermore, this depends on the number system used, so that with the binary system there is indeed only one way of producing the "new" number; however, the choice of the number base here is arbitrary, and by choosing a larger base the number of possibilities may be increased accordingly. Indeed, in general there seems to be no limit to the size of the base that is used, so that correspondingly there would be no limit to the number of putative "new" real numbers even from the limited set in the "diago-

nal" being considered.

3) Additionally, also for true completeness, the entries in the "diagonal" list may presumably be ordered in any way, each giving another number X from the diagonal process. Thus, with the listing of just three numbers in ¶2.4 above, this gives 3! = 6 ways of ordering the numbers, and 6 ways of producing the number X even from a single replacement method with the binary system (0 to 1, 1 to 0). In the general case, with n entries, there would be n! ways of ordering the list, and hence n! of the numbers X rather than just one.

4) Thus, when the CDA is applied in the specified manner, not just once, but exhaustively in these ways, this would give a list of valid derived real numbers X that becomes extensively longer than the chosen "diagonal" list itself, and potentially of unlimited size.

6. Application of the CDA to the Natural Numbers

1) Since the CDA must be presumed to be a general argument, it is useful to see the result if it is applied to the natural numbers themselves.

2) Again, as shown in Scheme 5, this is presented in two columns, with column $\mathbb{N}$ being the indexing column, and column $\mathbb{N}'$ being the natural numbers in the format as extended in this case to the right with sufficient zeroes to allow the CDM to be applied. In this case, the sequence "⋯" on the left indicates that the zeroes extend indefinitely to the left. The bold digits are again those to be used in the CDA.

$\mathbb{N}$	$\mathbb{N}'$
1	⋯000**1**
2	⋯00**0**2
3	⋯0**0**03
etc.	etc.

Scheme 5.

3) If the CDA is applied as before (¶2.5), then this leads as before to a putatively "new" natural number:

$$? \quad \cdots 0112 = Y$$

However, it is clear that this is simply another natural number ("one hundred and twelve"), to be found further down the list.

4) This test of the CDA on the natural numbers themselves leads to a contradiction, which again suggests that is some doubt in using the CDA in its more general applications (Hofstadter, 1979).

7. A One-to-One Correlation between the Natural Numbers and Real Numbers

1) The earlier discussion has cast some doubt on the conclusion conventionally drawn for the CDA, that is, that it is not possible to produce a one-to-one

correspondence between the natural numbers and the real numbers. However, from some clues provided by this discussion, particularly from the format of the natural numbers $\mathbb{N}'$ that has been used in ¶6.2, it turns out that it is possible to produce such a correspondence.

2) Consider, for example, the sequence shown in Scheme 6 that follows the development of the decimal expansion for the number π/10, where the digits of real-number entries in the B column (omitting 0.) are inverted to produce the corresponding natural-number entry in column A, leading therefore eventually to the pairing:

$$\begin{array}{ccc} \textbf{A} & & \textbf{B} \\ \cdots 951413 & \Leftrightarrow & 0.314159\cdots \end{array}$$

where the postfixed "⋯" on the right-hand side is to be read as the remainder of the decimal expansion of π/10, and that prefixed on the left-hand side is to be read as the mirror image of this expansion. Here the **A**-numbers may be considered as derived from the **B**-numbers either by reflecting them across the decimal point, or by rotating them by 180˚ about it.

A		B
3	⇔	0.3
13	⇔	0.31
413	⇔	0.314
1413	⇔	0.3141
etc.		etc.

Scheme 6.

3) It follows that the real-number entries in column **B** may be made as close as required to the decimal expansion of π/10, while the entries in column **A** still remain natural numbers. The limit of this process therefore provides a one-to-one correspondence between the two forms ($\mathbb{N} \Leftrightarrow \mathbb{R}$) as specified.

4) This may be extended to the general case, that is, for real numbers not limited to the range (0,1). In this case, the decimal point will have a natural number to the left of it, which for generality needs to be put with zeroes at its left end, and then the decimal part can be intercalated between the digits of this extended form of the natural number. For example, in the case of 10π as the real number, then writing this as ⋯00031.4159⋯, the derived natural number would be ⋯0**9**0**5**0**1**3**4**1 where the transferred decimal digits are put in bold for emphasis. The one-to-one correspondence, in this case, is thus:

$$\begin{array}{ccc} \mathbb{N} & \Leftrightarrow & \mathbb{R} \\ \cdots 090501341 & \Leftrightarrow & 31.4159\cdots \end{array}$$

5) Using now symbols rather than digits, this one-to-one correlation can be represented for the real numbers in the general case by

$$\begin{array}{ccc} \mathbb{N} & \Leftrightarrow & \mathbb{R} \\ \cdots a_3 A_3 a_2 A_2 a_1 A_1 & \Leftrightarrow & \cdots A_3 A_2 A_1 . a_1 a_2 a_3 \cdots \end{array}$$

where the *a*'s and *A*'s are the individual digits, and the same mirroring for the two forms of "⋯" is taken to apply.

8. Wilfred Hodges Strictures on "Hopeless Papers"

1) Looming above any discussion of the CDA must be the criticisms published by Wilfred Hodges (Hodges, 1998), in a review arising from his experiences with "hopeless papers" during his work as an editor and as a referee/reviewer; these criticisms were aimed specifically at the authors of submitted manuscripts, to rebut their objections to the CDA.

2) In this connection, he presented a version of the CDA that he apparently considers to be authoritative (albeit, as Hodges says, "not in Cantor's own words") (Hodges, 1998). Without going into his version in detail, the present two critiques of §4 still apply to his format. For the procedure again only considers the "new" number relative to those in the initial "diagonal" sequence with the later entries not having been examined, so that this does not rule out this number being in this latter part of the list-as noted already in ¶4.2 above. Furthermore, his specific choice of the mutation process (changing any "4" to a "5", but otherwise changing any other digit to a "4") will still give a specific sequence of digits from the range 0 - 9, which will therefore still be a real number, as noted already in ¶4.5 above.

3) Furthermore, none of his quoted papers (Hodges, 1998) seems to deal with the critiques presented earlier here.

9. The Term "Infinity"

1) The current approach to this term is encapsulated in a recent text dealing exhaustively with it from a diversity of aspects, historical, mathematical, philosophical, and religious; the text reveals its viewpoint in its title: "The Infinite" (Moore, 2018), thus taking the term to refer to a concrete entity.

2) However, this use of the term "infinity", and even the use of a symbol for it (∞), seems to imply that is a defined entity or quantity, that is, a number. This is arguably a simple category error, for the use of the alternative form of the word "infinity" as "endlessness" reveals more clearly that it refers to the *absence* of a feature to a process, that is, to the *absence of any limitation to the length of this process*.

3) The parallels here are such "absence" terms as "blackness" (the absence of light in the environment, or the absence of reflectivity for a surface), or "vacuum" (the absence of material content in a volume of space), or "silence" (the absence of sound) where again comparatives evidently cannot be applied, one material object cannot be "blacker" than another, one void cannot be more vacuous than another, one silence cannot be more silent than another. In the present context, such a viewpoint, indeed, provides a balm to both personal and philosophic anxieties in trying to grasp and face up to infinite sequences.

4) Otherwise, the consequences of viewing "infinity" as a defined quantity, is

presumably what one earlier author meant when he referred to this area as a "sematic quagmire" (Hogben, 1960).

10. Concluding Remarks

The present discussion has focused on the CDA as the prime source for considering the cardinalities of the natural numbers and the real numbers, and has necessarily not considered other publications by Cantor and later workers where this distinction is dealt with. However, in so far as the sources cited here represent the current consensus on the CDA, the present critiques suggest that the basis for consensus needs to be re-examined. In particular, they cast doubt on the formulation of "transfinite numbers", conventionally symbolised by the Hebrew letter $\aleph$ ("aleph"). They also cast doubt on the use of the CDA in the claimed impossibility of "proving" all mathematical propositions, originating with the mathematician Kurt Gödel (Hofstadter, 1979).

Conflicts of Interest

The author declares no conflicts of interest.

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