International Journal of Pathogen Research

5(3): 8-21, 2020; Article no.IJPR.61778
ISSN: 2582-3876

Modeling Coronavirus Spread Rate Utilizing Dimensional Analysis via an Irredundant Set of Fundamental Quantities

Mostafa Ali Rushdi[1,2] and Ali Muhammad Rushdi[3*]

[1]*Interdisciplinary Graduate School of Engineering Sciences (IGSES-ESST), Kyushu University, Fukuoka 816-8580, Japan.*
[2]*Faculty of Engineering and Technology, Future University in Egypt (FUE), New Cairo, 11835, Egypt.*
[3]*Department of Electrical and Computer Engineering, Faculty of Engineering, King Abdulaziz University, Jeddah, 21589, Saudi Arabia.*

Authors' contributions

This work was carried out in collaboration between the two authors. Author MAR envisioned and designed the study, proposed the use of an irredundant set of fundamental quantities, performed the analysis, participated in the literature search, solved the detailed examples and added the scalar exposition and explanations. Author AMR proposed performing dimensional analysis via the matrix method of Gauss-Jordan elimination and nullspace construction, wrote the preliminary manuscript, and managed the literature search. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/IJPR/2020/v5i330133
Editor(s):
(1) Dr. Ohanu, Ekechukwu Martin, University of Nigeria, Nigeria.
(2) Dr. Khadiga Ahmed Ismail Eltris, Ain Shams University, Egypt.
(3) Dr. Jasini A. Musa, University of Maiduguri, Nigeria.
Reviewers:
(1) Abhibhav Sharma, SC&SS Jawaharlal Nehru University, India.
(2) Samad Noeiaghdam, National Research Technical University, Russia.
(3) Arindam Kumar Paul, Khulna University, Bangladesh.
(4) Gratien Twaagirumukiza, INES Ruhengeri, Rwanda.
(5) Moien Ahmad Ansari, India.
Complete Peer review History: http://www.sdiarticle4.com/review-history/61778

Original Research Article

Received 05 August 2020
Accepted 11 October 2020
Published 04 November 2020

ABSTRACT

The phenomenon of spread of a (pathogenic) virus involves many physical variables, and is not amenable to satisfactory analysis via conventional methods. Dimensional Analysis (DA) is singled out as a simple and accessible way that can determine (at least qualitatively) how virus spread is related to seven physical quantities that are thought to influence it. However, classical DA deduces

**Corresponding author: Email: arushdi@yahoo.com, arushdi@kau.edu.sa;*

four dimensionless products only, none of which incorporates temperature and humidity, despite the obvious relevance of these two meteorological factors. This paper proposes an alternative version of dimensional analysis using a novel irredundant set of three fundamental quantities only. This new DA version produces five dimensionless products, four of which are essentially a replication of the old ones, while the fifth is a novel product that relates both humidity and temperature to other influencing factors. Our novel DA solution is a significant contribution, since it provides a more realistic model for virus spread rate, and it does not ignore any of the essential influencing factors. Such a model might lead to a better understanding of the determinants of spread for the novel coronavirus SARS-CoV-2 that causes the ongoing COVID-19 fatal pandemic.

Keywords: Dimensional analysis; dimensionless products; virus spread rate; irredundant set; COVID-19.

1. INTRODUCTION

Dimensional analysis (DA), also called the principle of similitude, is an effective way to analyze a physical phenomenon without explicit knowledge of its governing physical laws, provided we are confident that such laws really exist and apply to the pertinent phenomenon. Dimensional Analysis has a long history extending for several centuries [1,2], but it was almost one century ago that several pioneers, including Buckingham [3,4], Bridgman [5] and many others [6-18] laid the foundation for modern DA. Recently, DA applications have permeated diverse areas of medical disciplines [19-23], in general, and pathogenic research [24-28], in particular. Two recent papers [27,28] utilized Dimensional Analysis to model virus spread rate. The appearance of these papers is timely and opportune, since they constitute the first attempt to make use of DA in exploring the spread of the novel coronavirus (SARS-CoV-2). This virus causes the ongoing COVID-19 fatal pandemic, which is an unprecedented major threat for humanity [29].

The fundamental dimensions used in DA *may* be freely chosen, but they *should* be chosen *appropriately* for the specific problem at hand. Selection of too few fundamental quantities produces excessive ambiguities, and undermines the utilization of the dimensional homogeneity requirement in deriving and verifying formulas and in constructing dimensionless products [15]. Use of too many fundamental quantities causes redundancy and dependencies among the dimensional equations, which deprives the dimensional matrix of the full-rank characteristic [7,10,13,15,28,30,31].

The International System of Units (SI System) uses seven fundamental dimensions plus two supplementary ones. The SI system suffers from few ambiguities such as its assignment of non-distinct designations for torque and energy [15]. Usually, this system is rather convenient for handling typical standard problems of Dimensional Analysis. In many cases, it suffices to use only three to five of its reference dimensions, which constitute a subset of the $MLTI\theta$ set, comprising Mass (M), Length (L), Time (T), electric current (I), and Temperature (θ). An alternative recently-proposed dimensional system (called the CLT system) uses only three fundamental quantities, and attempts to eliminate redundancy in the $MLTI\theta$ system [32]. The CLT system is not superior to (and cannot replace) the $MLTI\theta$ system for a wide spectrum of problems, but it might occasionally produce more satisfactory results for a specific class of problems, in which redundancy is not desirable.

This paper addresses the problem of (pathogenic) virus spread rate using techniques of Dimensional Analysis. We explore and criticize the recently-published $MLTI\theta$-based DA solution [27] of this problem. We employ the CLT set to find an improved solution for the original problem, which does not ignore any of the variables deemed relevant from the outset. The improved solution essentially adds a new dimensionless product to four other products obtained by the old analysis. This new product accounts reasonably for both humidity and temperature, which were inadvertently ignored by the original analysis.

Throughout this paper, we consider that a sought product π_j of a set of physical variables is dimensionless if, and only if, the exponents of these variables are a solution of the set of p homogeneous linear equations (not necessarily linearly independent) in n unknowns, expressed in matrix form as [7,10,13,15,28,30,31]:

$$Dz = 0, \quad (1)$$

where D is the $p \times n$ dimensional matrix. This matrix has p rows which represent the adopted fundamental reference dimensions or elements of the dimensional basis, and n columns, which denote the variable exponents in the sought dimensionless product or, with a gross (albeit common and appealing) abuse of notation, designate the physical variables themselves. We will designate a column twice: (a) by the correct exponent notation, and (b) by the common variable notation. A typical entry of this matrix is the exponent to which a reference dimension (row) is raised in the dimensional product formula representing the particular variable (column). The vector $\boldsymbol{z}$ comprises the n variable exponents in the sought dimensionless product, which are unknown constants, yet to be inter-related (partially determined). The Gauss-Jordan algorithm [28,33] achieves the purpose of inter-relating the exponents by applying elementary row operations that transform the matrix $\boldsymbol{D}$ into a reduced row echelon form (RREF), as shown in forthcoming Tables 1, 2, and 4. The vector of indices $\boldsymbol{z}$ is not written as a column vector to the right of the dimensional matrix as suggested by Eq. (1), but is written (in a non-conventional way) as a row vector on top of it [28,30,31,34]. In addition, the equality sign in Eq. (1) is omitted and implicitly understood, while the zero vector in the R.H.S. of Eq. (1) is added as an extra vector for $\boldsymbol{D}$ resulting in an *augmented matrix*, to whose entire rows we apply the same elementary row operations. Such operations are explained by assignment operations written in the leftmost column, where $E_i^{(k)}$ denotes the equation of row i at stage or tableau k. Further information about the Gauss-Jordan algorithm might be found in standard texts on numerical analysis [33], and its application in DA problems is exposed and demonstrated in several recent papers [28,30,31] and in some references therein. The reader might also consult Appendix A for a scalar interpretation of our matrix solution.

The remainder of this paper is structured as follows. Section 2 introduces the irredundant CLT dimensional basis, while Section 3 compares it to the standard $MLTI\theta$ basis. Sections 4 and 5 solve the problem of virus spread rate, first via the $MLTI\theta$ basis and then via the CLT basis. Detailed discussions are given in these two sections to compare the DA solutions to the current observations about the meteorological factors influencing the spread rate of coronaviruses. Section 6 exposes the most important limitations ascribed (correctly or incorrectly) to Dimensional Analysis in general, stresses those limitations that apply to the current work, and points out potential ways to mitigate such limitations. Section 7 concludes the paper. The paper is also supplemented with an appendix that presents a scalar formulation and solution of the main DA problem.

2. THE IRREDUNDANT CLT DIMENSIONAL BASIS

Dimensional Analysis is mostly based on the use of the redundant $MLTI\theta$ multidimensional system [15,28]. An alternative system using only three fundamental dimensions is the CLT system, where C stands for 'Chakr,' a transliteration of the Hindi word for wheel [32]. If an arbitrary physical quantity Q is expressed in the $MLTI\theta$ and CLT bases by the vectors of indices $\boldsymbol{r} = [r_1\ \ r_2\ \ r_3\ \ r_4\ \ r_5]^T$ and $\boldsymbol{R} = [R_1\ \ R_2\ \ R_3]^T$. Hence, the dimension $[Q]$ of Q is given by [32]

$$[Q] = M^{r_1}\, L^{r_2}\, T^{r_3}\, I^{r_4}\, \theta^{r_5} = C^{R_1}\, L^{R_2}\, T^{R_3}$$
$$= (C^1\, L^{-3}\, T^3)^{r_1}\, L^{r_2}\, T^{r_3}\, (C^{0.5}\, L^2\, T^{-1})^{r_4}\, (C^1\, L^{-1}\, T^1)^{r_5}.$$

Consequently, the various $\boldsymbol{R}$ indices are related to the $\boldsymbol{r}$ indices by

$$R_1 = r_1 + 0.5\, r_4 + r_5, \quad R_2 = -3r_1 + r_2 + 2r_4 - r_5, \quad R_3 = 3r_1 + r_3 - r_4 + r_5\,,$$

Hence, the vectors of indices are related by the transformation $\boldsymbol{R} = \boldsymbol{T}\,\boldsymbol{r}$, or explicitly as

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0.5 & 1 \\ -3 & 1 & 0 & 2 & -1 \\ 3 & 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix}. \quad (2)$$

The five vectors comprising the 3×5 transformation matrix $\boldsymbol{T}$ are the vectors of exponents for the variables $M, L, T, I,$ and θ in the CLT basis. For example, If $Q = M$ then $\boldsymbol{r} = [1\ \ 0\ \ 0\ \ 0\ \ 0]^T$ and $\boldsymbol{R} = [1\ \ -3\ \ 3]^T$, i.e., $M = C\, L^{-3}\, T^3$. The CLT basis has many important features, including the following ones [32], which might be viewed as advantages only in certain favorable situations:

1. Mass is defined as a derived quantity (CT^3) per unit volume, in agreement with the observation that any particle of non-zero mass is of non-zero volume also.
2. The dimensions of permittivity, permeability, and impedance do not include a Mass element, but solely depend

on Length and Time. This makes sense for free-space quantities.

3. Temperature and energy have the same dimensions, and Boltzmann constant is dimensionless.

Feature 3 above is the real cause of the anticipated success of the CLT -based dimensional analysis in assessing the virus spread rate. In fact, if the $MLTI\theta$ is arbitrarily endowed with this feature, it can be made to achieve a similar success and attain the same result [28]. However, this same feature might be detrimental in other situations, in which irredundancy is a shortcoming rather than a merit. For example, consider $MLTI\theta$-based and CLT-based DA derivation of the physical relation between energy, temperature, and Boltzmann constant. The $MLTI\theta$ -based solution elegantly and exactly obtains the correct relation (up to an arbitrary proportionality constant), while the CLT-based solution fails to identify this relation explicitly as it obtains two dimensionless products, which cannot be related solely by DA means.

3. A COMPARISON BETWEEN THE $MLTI\theta$ and CLT DIMENSIONAL BASES

As stated earlier, any sought product π_j of a set of physical variables is dimensionless if, and only if, the exponents of these variables are a solution of (1), which constitutes a set of p homogeneous linear equations in n unknowns. The $p \times n$ dimensional matrix $\boldsymbol{D}$ in (1) has p rows, which represent the adopted fundamental reference dimensions ($p = 5$ ($MLTI\theta$) or 3 (CLT)), and n columns, which denote the variable exponents in π_j . The 3×5 transformation matrix $\boldsymbol{T}$ that appears in (2) can be left multiplied with the $MLTI\theta$ -based $5 \times n$ dimensional matrix to produce the CLT-based $3 \times n$ dimensional matrix. We reiterate that we will designate a column by both the corresponding exponent and the corresponding variable.

We will now solve a standard DA problem using each of the afore-mentioned two bases. Our purpose is to demonstrate the steps of the modern DA matrix method [15,28,30,31] and to show the equivalence and subtle differences between solutions in terms of the two different bases. Consider the situation of an infinite straight wire carrying a constant current I placed in a medium of permeability μ perpendicular to a uniform magnetic field of intensity H and flux density B. The force per unit length F affecting the wire is required, and hence the variable F that must be a regime variable is placed last in a proposed dimensionless product $\pi = k\ \mu^m\ H^h\ I^i\ B^b\ F^f$, where k is a dimensionless constant, and no temperature dimension is involved. Tables 1 and 2 demonstrate the Gauss-Jordan procedure [28,30,31] for solving this problem in the $MLTI$ and CLT dimensional bases, respectively. The same final solution is obtained in both tables. However, the $MLTI$-based solution is obviously more efficient, has fewer tableaus and simpler numbers, and hence it is less error prone than the CLT-based solution. The Gauss-Jordan procedure detects redundancy in the $MLTI$-based dimensional matrix, as it creates an

Table 1. The Gauss-Jordan procedure for determining the force per unit length exerted on an infinite wire in a uniform magnetic field using the $MLTI$ basis

	m μ	h H	i I	b B	f F	
$E_1^{(0)}$	1	0	0	1	1	0
$E_2^{(0)}$	1	−1	0	0	0	0
$E_3^{(0)}$	−2	0	0	−2	−2	0
$E_4^{(0)}$	−2	1	1	−1	0	0
$E_1^{(1)} \leftarrow E_1^{(0)}$	1	0	0	1	1	0
$E_2^{(1)} \leftarrow E_2^{(0)} - E_1^{(0)}$	0	−1	0	−1	−1	0
$E_3^{(1)} \leftarrow E_3^{(0)} + 2\,E_1^{(0)}$	0	0	0	0	0	0
$E_4^{(1)} \leftarrow E_4^{(0)} + 2\,E_1^{(0)}$	0	1	1	1	2	0
$E_1^{(2)} \leftarrow E_1^{(1)}$	1	0	0	1	1	0
$E_2^{(2)} \leftarrow -E_2^{(1)}$	0	1	0	1	1	0
$E_4^{(2)} \leftarrow E_4^{(1)} + E_2^{(1)}$	0	0	1	0	1	0
π_1	−1	−1	0	1	0	
π_2	−1	−1	−1	0	1	

Table 2. The Gauss-Jordan procedure for determining the force per unit length exerted on an infinite wire in a uniform magnetic field using the CLT basis

	m μ	h H	i I	b B	f F	
$E_1^{(0)}$	0	0.5	0.5	0.5	1	0
$E_2^{(0)}$	−6	1	2	−5	−3	0
$E_3^{(0)}$	3	−1	−1	2	1	0
$E_1^{(1)} \leftarrow E_1^{(0)} - 0.5\, E_2^{(0)}$	3	0	−0.5	3	2.5	0
$E_2^{(1)} \leftarrow E_2^{(0)}$	−6	1	2	−5	−3	0
$E_3^{(1)} \leftarrow E_3^{(0)} + E_2^{(0)}$	−3	0	1	−3	−2	0
$E_1^{(2)} \leftarrow E_1^{(1)} + 0.5\, E_3^{(1)}$	1.5	0	0	1.5	1.5	0
$E_2^{(2)} \leftarrow E_2^{(1)} - 2\, E_3^{(1)}$	0	1	0	1	1	0
$E_3^{(2)} \leftarrow E_3^{(1)}$	−3	0	1	−3	−2	0
$E_1^{(3)} \leftarrow E_1^{(2)}/(1.5)$	1	0	0	1	1	0
$E_2^{(3)} \leftarrow E_2^{(2)}$	0	1	0	1	1	0
$E_4^{(3)} \leftarrow E_4^{(2)} + 2\, E_1^{(2)}$	0	0	1	0	1	0
π_1	−1	−1	0	1	0	
π_2	−1	−1	−1	0	1	

all-0 row in the second tableau of Table 1 (that we conveniently delete in subsequent tableaus). The CLT-based solution does not encounter a row whose entries are all 0. Therefore, each of the $MLTI$-based and CLT-based matrices has a rank r of 3, and a defect $(n-r)$ of 2. At the last stage of each solution, the $p \times n$ dimensional matrix $\boldsymbol{D}$ becomes an $r \times n$ matrix that is partitioned into an $r \times r$ unit matrix and an $r \times (n-r)$ matrix $\boldsymbol{C}$. We now construct a full-rank $(n-r) \times n$ matrix $\boldsymbol{K}$ that is partitioned into two matrices: the negative transpose $-\boldsymbol{C}^T$ of $\boldsymbol{C}$, left juxtapositioned to an $(n-r) \times (n-r)$ unit matrix. The $(n-r) = 2$ rows of $\boldsymbol{K}$ depict a complete set of the dimensionless products as shown at the bottom of each of Tables 1 or 2. These are: $\pi_1 = B/\mu H$ and $\pi_2 = F/\mu HI$, which constitute a complete set of dimensionless products. It is known (outside the scope of dimensional analysis) that each of these two products is a constant equal to 1, and hence $B = \mu H$ and $F = \mu HI = BI$.

4. DERIVATION OF THE VIRUS SPREAD RATE VIA THE $MLTI\theta$ DIMENSIONAL BASIS

Contreras et al. [27] utilized DA to derive all dimensionless products involving the eight quantities considered in Table 3. These quantities include the required output (virus spread rate) besides seven determining or influencing variables. The selection of these eight quantities is based on detailed plausible reasoning reported in [27], and they apparently satisfy the conditions of the Buckingham Pi Theorem [3,15,30]. Since none of these eight quantities is of electromagnetic nature, the $MLTI\theta$ dimensional basis is conveniently replaced in [27] by a non-electromagnetic subset of it, the $MLT\theta$ basis. Each dimensionless product of the set of eight quantities in Table 3 is now expressed in the product form

$$\pi = k \; H^a \; P_r^{\,b} \; c_a^{\,c} \; \theta^d \; V_P^{\,e} \; C_e^{\,f} \; E_{fs}^{\,g} \; I_P^{\,h}, \qquad (3)$$

where k is a dimensionless constant, while a, b, c, d, e, f, g, and h are unknown exponents to be inter-related. Employing classical determinant-based DA techniques, Contreras et al. [27] proved that π is given by

$$\pi = k \; \left(V_P P_r^{\,2}/c_a\right)^e \; \left(C_e\, c_a/\, P_r^{\,3}\right)^f \; \left(E_{fs}\, P_r^{\,2}\right)^g \, (I_P)^h \; = k \; (\pi_1)^e \; (\pi_2)^f \, (\pi_3)^g \, (\pi_4)^h, \qquad (4)$$

where $\pi_1 = V_P P_r^{\,2}/c_a$, $\pi_2 = C_e\, c_a/\, P_r^{\,3}$. $\pi_3 = E_{fs}\, P_r^{\,2}$, and $\pi_4 = I_P$ constitute a complete set of dimensional products [28]. According to Buckingham Pi theorem, these products satisfy:

$$\Phi\,(\pi_1\,, \pi_2 . \pi_3\,, \pi_4) = 0. \qquad (5)$$

The virus spread rate V_P can be mathematically modeled by expressing its regime π_1 as an arbitrary function Ψ (to be determined experimentally) of the other three regimes, namely

$$\pi_1 = \Psi\,(\pi_2\,, \pi_3\,, \pi_4), \qquad (6)$$

or explicitly as

$$V_P \;\; = (c_a \,/\, P_r^{\,2}) \;\; \Psi\,((\,C_e\, c_a/\, P_r^{\,3})\,,\, E_{fs}\, P_r^{\,2}\,, I_P). \; (6a)$$

Table 3. Symbols, units and dimensions of eight variables for virus transmission speed

No	Variable	Symbol	$MLT\theta$ dimensions	CLT dimensions
1	Absolute air humidity	H	$M^1\ L^{-3}\ T^0\ \theta^0$	$C^1\ L^{-6}\ T^3$
2	Precipitation (rainfall)	P_r	$M^0\ L^1\ T^0\ \theta^0$	$C^0\ L^1\ T^0$
3	Airflow (air current related to ventilation processes)	c_a	$M^0\ L^3\ T^{-1}\ \theta^0$	$C^0\ L^0\ T^1$
4	Ambient temperature	θ	$M^0\ L^0\ T^0\ \theta^1$	$C^1\ L^{-1}\ T^1$
5	Virus spread rate	V_P	$M^0\ L^1\ T^{-1}\ \theta^0$	$C^0\ L^1\ T^{-1}$
6	Period of seasonal changes in behavior	C_e	$M^0\ L^0\ T^1\ \theta^0$	$C^0\ L^3\ T^{-1}$
7	Population area density	E_{fs}	$M^0\ L^{-2}\ T^0\ \theta^0$	$C^0\ L^{-2}\ T^0$
8	Pre-existing immunity	I_P	$M^0\ L^0\ T^0\ \theta^0$	$C^0\ L^0\ T^0$

Contreras et al. [27] recognized that the parameters of temperature and humidity fail to appear in their model, and claimed that this does not mean that these parameters are "not relevant in reality, but that they could be less important with respect to the rest of the parameters chosen in the system under study." We stress that these two parameters are relevant in reality, but we totally disagree with the assertion that they are definitely of less importance compared with the remaining parameters used. It is no surprise that $MLT\theta$ -based analysis fails to find a role for temperature (being the only variable in the model with a θ dimension) or for humidity (being the only variable in the model with an M dimension). By contrast, CLT-based analysis can recognize or capture the anticipated relevance of both temperature and humidity.

We stress that most papers surveyed stress that both temperature and humidity are of some relevance (that cannot be neglected) to the spread rate for coronaviruses, in general (and for the novel coronavirus, in particular) [38-49]. A notable exception to the general trend in the papers surveyed is the work of Jamil et al. [50], who tested the hypothesis that COVID-19 spread is temperature-dependent using data derived from nations across the world and provinces in China, and found no evidence of a pattern between spread rates and ambient temperature, suggesting that the SARS-CoV-2 is unlikely to behave as a seasonal respiratory virus. Abdollahi and Rahbaralam [51] found that there is a moderate inverse correlation between temperature and the daily number of infections. Briz-Redón and Serrano-Aroca [52] suggest that 'the disparate findings reported seem to indicate that the estimated impact of hot weather on the transmission risk is not large enough to control the pandemic.'

5. DERIVATION OF THE VIRUS SPREAD RATE VIA THE CLT DIMENSIONAL BASIS

Table 4 displays the stages of the procedure of Gauss-Jordan elimination for solving the problem of the virus spread rate in the CLT dimensional basis, and its bottom is a matrix representation of the dimensionless products obtained for this problem [15,28], which turn out to be five rather than four in the present case. The general expression for a dimensionless product π is now given by

$$\pi = k\ \left(\theta\, C_e{}^2 / P_r{}^5 H\right)^d\ \left(V_P\ C_e / P_r\right)^e\ \left(C_e\, c_a /\right.$$
Pr3c Efs Pr2g IPh =k π0 d π1(n)e π2c π3g
$$(\pi_4)^h, \qquad (7)$$

where $\pi_0 = \theta\, C_e{}^2 / P_r{}^5 H$, $\pi_1{}^{(n)} = V_P\ C_e / P_r$, $\pi_2 = C_e\, c_a / P_r{}^3$, $\pi_3 = E_{fs}\, P_r{}^2$, and $\pi_4 = I_P$ constitute a new complete set of dimensional products. This new set has an extra totally new product π_0, and it retains (exactly) three of the products in Section 4 (π_2, π_3 and π_4). It has a dimensionless product $\pi_1{}^{(n)}$ that can be obtained as a composite product of the original set ($\pi_1{}^{(n)} = \pi_1\, \pi_2$), and again serves as a regime for the variable of interest V_P . According to Buckingham Pi Theorem, these dimensionless products satisfy:

$$\Phi^{(n)}\ (\pi_0,\ \pi_1{}^{(n)}\ , \pi_2 . \pi_3\ , \pi_4) = 0. \qquad (8)$$

The virus spread rate V_P can be modeled by expressing its regime $\pi_1{}^{(n)}$ as an arbitrary function $\Psi^{(n)}$ (to be determined experimentally) of the other four regimes, namely

$$\pi_1{}^{(n)} = \Psi^{(n)}\ (\pi_0, \pi_2\ , \pi_3\ , \pi_4), \qquad (9)$$

Table 4. Gauss-Jordan elimination for the virus spread problem in the CLT basis

	$\boldsymbol{a}$ $\boldsymbol{H}$	$\boldsymbol{b}$ $\boldsymbol{P_r}$	$\boldsymbol{f}$ $\boldsymbol{C_e}$	$\boldsymbol{d}$ $\boldsymbol{\theta}$	$\boldsymbol{e}$ $\boldsymbol{V_P}$	$\boldsymbol{c}$ $\boldsymbol{c_a}$	$\boldsymbol{g}$ $\boldsymbol{E_{fs}}$	$\boldsymbol{h}$ $\boldsymbol{I_P}$	
${E_1}^{(0)}$	1	0	0	1	0	0	0	0	0
${E_2}^{(0)}$	−6	1	0	−1	1	3	−2	0	0
${E_3}^{(0)}$	3	0	1	1	−1	−1	0	0	0
${E_1}^{(1)} \leftarrow {E_1}^{(0)}$	1	0	0	1	0	0	0	0	0
${E_2}^{(1)} \leftarrow {E_2}^{(0)} + 6\,{E_1}^{(0)}$	0	1	0	5	1	3	−2	0	0
${E_3}^{(1)} \leftarrow {E_3}^{(0)} - 3\,{E_1}^{(0)}$	0	0	1	−2	−1	−1	0	0	0
π_0	−1	−5	2	1	0	0	0	0	
${\pi_1}^{(n)}$	0	−1	1	0	1	0	0	0	
π_2	0	−3	1	0	0	1	0	0	
π_3	0	2	0	0	0	0	1	0	
π_4	0	0	0	0	0	0	0	1	

This arbitrary function can be written as a quadruply infinite series comprising terms of the general monomial form $k_{qrst}\,{\pi_0}^q\,{\pi_2}^r\;\;{\pi_3}^s\;\;{\pi_4}^t$ where $q, r, s,$ and t take all possible positive, zero, or negative values. Equation (9) might be written explicitly as

$$V_P = (P_r / C_e)\quad \Psi^{(n)}\left(\left(\theta\,{C_e}^2 / {P_r}^5 H\right), (\,C_e\,c_a / {P_r}^3)\,,\, E_{fs}\,{P_r}^2\,, I_P\right). \tag{9a}$$

The new model for the virus spread rate V_P in (9a) retains most of the features of the earlier model in [27] (reproduced in (6a)), but it reasonably accounts for the two parameters of temperature θ and humidity H , which were inadvertently excluded in [27]. Equation (9a), however stresses that temperature and humidity are of secondary relevance to the virus spread rate, since they are inter-regime influences, rather than intra-regime ones. The very existence of this relevance (together with its being of limited or secondary nature only) seems to be in line with the reported disparity among the cited references [38-52]. Qualitative reasoning [30,35-37] show that, according to (9a), temperature and humidity have opposing influences on virus spread rate. In fact, the virus spread rate is invariant to the ratio of temperature to humidity according to (9a). This finding might be changed if Table 3 is found to be a non-exhaustive one, i.e., if pathogenic researchers can ascertain that some pertinent physical variables are missing in this table.

We stress that the model in [27] is simply out of competition according to Occam's razor [53-56] since it fails to capture all essential features. The present model looks definitely more satisfactory than the earlier one, but it cannot be deemed totally acceptable until it can be shown to encompass all relevant influencing variables, and moreover to fit ample experimental observations. We have started working on that issue by surveying published data about the determinants of virus spread rate. Obviously, such data is expected to be species-specific, and though several types of viruses have caused serious diseases, epidemics, and pandemics lately, we are more interested herein in studying the novel coronavirus that causes the ongoing COVID-19 fatal pandemic.

6. ON LIMITATIONS OF DIMENSIONAL ANALYSIS

Utilization of Dimensional Analysis in the scientific literature has been undermined due to its genuine limitations, as well as due to many past and present fallacious misconceptions about it. In the following, we discuss the most important limitations ascribed (correctly or incorrectly) to Dimensional Analysis in general [1,15,28,30,57]. We stress those limitations that apply to our current work, and point out ways, if any, that serve to mitigate such limitations.

- Dimensional Analysis might only lead to preliminary or first-cut understanding of the solution of a physical problem. It might yield information about the number of variables that are necessary to describe such a problem, and it can reduce the number of variables, which were initially supposed (without confirming full evidence) to influence a given physical problem. However, DA does not serve as a totally-independent and complete method in many situations. Therefore, it should be used carefully and appropriately, with assistance prudently sought from sources outside its scope. For example, our current DA solution is just an initial attempt that might be improved through the incorporation of more influencing variables. This solution is to be checked, partially modified or even totally abandoned through extensive experimentation. In any case, this solution might be credited for setting the stage for potentially-fruitful future experiments or measurements.

- The initial consideration of the explored problem is a task assigned to the unaided researcher and is out of the scope of DA, which obviously cannot invent variables and can only deal with the variables supplied to it [1,28]. As with any method of analysis, DA depends on being furnished at the outset with all the physical explanatory variables affecting the situation. If one fails to include an important effect, the model will attempt to compensate but it cannot replace real genuine knowledge. We will always need some understanding of the effects that are in operation. Note that over-specification might not be as harmful as under-specification. An incomplete set of independent quantities (basis (influencing) and regime (influenced) ones) may effectively destroy the analysis. In fact, a single error of variable omission might be catastrophically detrimental to the analysis [57]. Superfluous independent variables never destroy the analysis, but they rob it of its power and complicate the result unnecessarily. Theoretically, every superfluous independent variable (included in the set of independent variables) inadvertently augments the DA solution with a superfluous dimensionless product [57]. However, such a superfluous product might be eliminated (but only after due payment in extra effort caused by initial lack of insight) if there exists a broad range of conditions where the superfluous product has no appreciable effect on the regime variable of interest. For our current DA analysis, we cannot rule out the possibility of under-specification, over-specification, or a mixture thereof. In the case of under-specification, the present DA is expected to be still reasonably explanatory with only some secondary effects missing. A case of over-specification is likely to be fixed via experimentation. Sonin [57] asserts that “completeness in the set of independent variables is not an absolute matter, but depends on how we choose to circumscribe the problem.’ For our current problem, scientific experience was utilized [27,28] to argue that the regime variable of virus spread rate should be influenced by a set of seven other variables such that the eight variables together form a complete set of independent variables. Then, the following question is posed: Assuming tentatively that this argument is correct, what does Dimensional Analysis tell us?
- Dimensional Analysis is ‘most useful in areas in which knowledge is developing through an intermediate stage at which the basic laws are already known, but there is still a lack of powerful methods of solution [1].’ It should not be applied to situations in which even the physical laws involved are totally unknown, i.e., such that one cannot secure the condition necessitated by Buckingham Pi Theorem as a pre-requisite of the validity of its result [30]. It seems safe to assert that DA is applicable in our current problem [27,28].
- Dimensional Analysis does not usually indicate whether important physical quantities or variables are missing from a proposed solution, and hence it cannot tell whether the correct set of physical variables is the one used. It is only when inconsistencies occur, that DA might give us a clue that something important has been omitted. Therefore, it might be advisable to include as many variables as possible in the dimensional analysis process. Such advice is really prudent, since over-specification (in contrast to under-specification) is neither catastrophic nor really irreversible.
- Dimensional Analysis can be applied only to systems involving physical quantities, several dimensions, and with positive, continuous variables. It does not constitute a complete mathematical analysis that provides numerical values. These have to be determined externally by experimentation, regression or data analysis, once DA modeling has been completed. Dimensional Analysis always needs to be supplemented or aided by other techniques, or, occasionally, by educated guessing or even by brilliant insight. This is certainly the case in the

present DA problem, for which ample and non-controversial data is still being collected.

- Dimensional Analysis naturally seeks a complete set of dimensionless products for the problem under consideration. This is a set, in which products are independent of one another, and such that any product (of some of the original dimensional variables) that does not belong to the set, can be expressed as a product of powers of the dimensionless products of the set. A DA limitation is that a complete set of dimensionless products is not unique. In fact, there are many sets that can play the role of a complete set, according to the way of partitioning variables to basis and regime variables. To limit the number of such sets in our present problem, we considered only a complete set that uses the variable of virus spread rate as a regime variable. It is also possible to enumerate all similar sets [30,31].
- Contrary to widespread belief (as evidenced by statements in a huge number of Internet sites), DA is capable of handling arbitrary functions (such as the ones in (8) and (9)), which might be even be transcendental functions (such as sinusoidal, hyperbolic, logarithmic, and exponential functions). However, it is outside the scope of DA to guess the nature of such functions or to do curve fitting for determining their parameter. It is true that DA-derivation of models involving transcendental functions is quite difficult, but it is not totally impossible. A case in point is the derivation by Middendorf [10] of the exponential relaxation curve of the transient electric current in an RC-circuit. Admittedly, such a derivation is highly dependent on the expertise of the researcher and on his ability to make some educated guess about the nature of the pertinent function.

7. CONCLUSIONS

This paper addressed the problem of (pathogenic) virus spread rate using DA techniques. The DA solution of this problem might contribute to a better understanding of the determinants of spread for the novel coronavirus that causes the ongoing COVID-19 fatal pandemic. We explored a solution of this DA problem that has been recently published and criticized the model obtained. We employed a recently proposed irredundant dimensional set to find an improved and novel solution that did not ignore any of the variables deemed relevant from the outset.

Future work might be suggested in a variety of possible directions. An immediate sequel of the present work might cast equation (9a) in an explicit form by using curve fitting to experimental data [58-60]. Another direction of research is to apply Dimensional Analysis in epidemiological studies that mainly involve differential equations [61-65] and to the transmission or spread of pathogenic viruses (other than the present coronavirus one) such as the human immunodeficiency virus (HIV) [66], influenza virus [67], dengue virus [68], hepatitis C virus [69], Zika virus [70] and yellow fever virus [71]. Other interesting areas of research involve the comparison of the spread of pathogenic viruses and computer viruses [72-74] and supporting pathogenic research with other engineering tools such as the Karnaugh map [75] and probabilistic methods [76-81].

COMPETING INTERESTS

The authors have declared that no competing interests exist.

REFERENCES

1. Macagno EO. Historico-critical review of dimensional analysis. Journal of the Franklin Institute. 1971;292(6):391-402.
2. Martins RD. The origin of dimensional analysis. Journal of the Franklin Institute. 1981;311(5):331-337.
3. Buckingham E. On physically similar systems; Illustrations of the use of dimensional equations. Physical Review. 1914;4(4):345-376.
4. Buckingham E. The principle of similitude. Nature. 1915;96(2405):396-397.
5. Bridgman PW. Dimensional analysis. Yale University Press, New Haven, CT, USA; 1922.
6. László A. Systematization of dimensionless quantities by group theory. International Journal of Heat and Mass Transfer. 1964;7(4):423-430.
7. Chen WK. Algebraic theory of dimensional analysis. Journal of the Franklin Institute. 1971;292(6):403-422.
8. Happ WW. Dimensional analysis via directed graphs. Journal of the Franklin Institute. 1971;292(6):527-533.

9. Görtler H. Dimensionsanalyse: Theorie der physikalischen Dimensionen mit Anwendungen (Dimensional analysis: The theory of physical dimensions with applications). Springer Berlin; 1975.
10. Middendorf WH. The use of dimensional analysis in present day design environment. IEEE Transactions on Education. 1986;E-29(4):190-195.
11. Hertkorn P, Rudolph S. Dimensional analysis in case-based reasoning. In the Proceedings of the International Workshop on Similarity Methods. 1998;163-178.
12. Brennan SN. On size and control: The use of dimensional analysis in controller design. Doctoral Dissertation, University of Illinois at Urbana-Champaign, Illinois, USA; 2002.
13. Hutter K, Jöhnk K. Theoretical foundation of dimensional analysis. In Hutter K, et al., (Editors), Continuum Methods of Physical Modeling, Springer Berlin Heidelberg. 2004;339-392.
14. Sonin AA. A generalization of the Π-theorem and dimensional analysis. Proceedings of the National Academy of Sciences. 2004;101(23):8525-8526.
15. Szirtes T. Applied dimensional analysis and modeling. Second Edition, Butterworth Heinemann, Burlington, MA, USA; 2007.
16. Hart GW. Multidimensional analysis: Algebras and systems for science and engineering. Springer Science & Business Media; 2012.
17. Pawlowski J. Die Ähnlichkeitstheorie in der physikalisch-technischen Forschung: Grundlagen und Anwendung (The Similarity Theory in Physical-Technical Research: Fundamentals and Application). Springer-Verlag; 2013.
18. Constantine PG, del Rosario Z, Iaccarino G. Data-driven dimensional analysis: Algorithms for unique and relevant dimensionless groups. arXiv preprint arXiv:1708.04303; 2017. Available:https://arxiv.org/pdf/1708.04303.pdf
19. FitzHugh R. Dimensional analysis of nerve models. Journal of Theoretical Biology. 1973;40(3):517-541.
20. Gunther B. Dimensional analysis and theory of biological similarity. Physiological Reviews. 1975;55(4):659-699.
21. Rácz Z. Examples of unconventional dimensional analysis. Acta Physica Hungarica. 1992;72(2-4):249-252.
22. Passariello F. Dimensional analysis in the venous system. Journal of Theoretical and Applied Vascular Research (JTAVR): A Journal of Research on Vascular Diseases. 2017;2(3):123-130.
23. Ganan-Calvo AM, Ramos JAH. The fractal time growth of COVID-19 pandemic: An accurate self-similar model and urgent conclusions. arXiv preprint arXiv:2003.14284; 2020. Available:https://arxiv.org/pdf/2003.14284.pdf
24. Kowalski WJ, Bahnfleth WP, Rosenberger JL. Dimensional analysis of UVGI air disinfection systems. HVAC&R Research. 2003;9(3):347-363.
25. Anekal SG, Zhu Y, Graham MD, Yin J. Dynamics of virus spread in the presence of fluid flow. Integrative Biology. 2009;1(11-12):664-671.
26. Burzio E, Bersani F, Caridi GC, Vesipa R, Ridolfi L, Manes C. Water disinfection by orifice-induced hydrodynamic cavitation. Ultrasonics Sonochemistry. 2020;60: 104740.
27. Contreras GS, Mora MR, Gómez PJ. Speed of virus infection by classical dimensional analysis. Contemporary Engineering Sciences. 2020;13(1):131-147.
28. Rushdi MA, Rushdi AM. Modeling virus spread rate via modern techniques of dimensional analysis. Journal of King Abdulaziz University: Computing and Information Technology. 2020;9(2):47-66.
29. Rushdi AM, Serag HA. Solutions of ternary problems of conditional probability with applications to mathematical epidemiology and the COVID-19 pandemic. International Journal of Mathematical, Engineering and Management Sciences. 2020;5(5):787-811.
30. Rushdi MA, Rushdi AM. On the fundamental masses derivable by dimensional analysis. Journal of King Abdulaziz University: Engineering Sciences. 2016;27(1):35-42.
31. Oladigbolu JO, Rushdi AM. Investigation of the corona discharge problem based on different computational approaches of dimensional analysis. Journal of Engineering Research and Reports. 2020;15(3):17-36.
32. Abubakr M. On dimensional analysis, redundancy in set of fundamental quantities and proposal of a new set. arXiv preprint arXiv:0710.3483; 2007.

Available:https://arxiv.org/ftp/arxiv/papers/0710/0710.3483.pdf
33. Burden RL, Douglas Faires J. Numerical analysis. Ninth Edition, Brooks/Cole, Boston, MA, USA; 2011.
34. Rushdi AM. Development of modified nodal analysis into a pedagogical tool. IEEE Transactions on Education. 1985; E-28(1):17-25.
35. Bhaskar R, Nigam A. Qualitative physics using dimensional analysis. Artificial Intelligence. 1990;45(1):73-111.
36. Kalagnanam J, Henrion M, Subrahmanian E. The scope of dimensional analysis in qualitative reasoning. Computational Intelligence. 1994;10(2):117-133.
37. Roque WL, Dos Santos RP. Qualitative reasoning, dimensional analysis and computer algebra. In Proceedings of the 1991 International Symposium on Symbolic and Algebraic Computation. 1991;460-461.
38. Lamarre A, Talbot PJ. Effect of pH and temperature on the infectivity of human coronavirus 229E. Canadian Journal of Microbiology. 1989;35(10):972-974.
39. Casanova LM, Jeon S, Rutala WA, Weber DJ, Sobsey MD. Effects of air temperature and relative humidity on coronavirus survival on surfaces. Applied and Environmental Microbiology. 2010;76(9): 2712-2717.
40. Chan KH, Peiris JM, Lam SY, Poon LL, Yuen KY, Seto WH. The effects of temperature and relative humidity on the viability of the SARS coronavirus. Advances in Virology; 2011. Article ID: 734690; 1-7.
41. Xie J, Zhu Y. Association between ambient temperature and COVID-19 infection in 122 cities from China. Science of the Total Environment. 2020;724:138201.
42. Tosepu R, Gunawan J, Effendy DS, Lestari H, Bahar H, Asfian P. Correlation between weather and Covid-19 pandemic in Jakarta, Indonesia. Science of the Total Environment. 2020;725:138436.
43. Prata DN, Rodrigues W, Bermejo PH. Temperature significantly changes COVID-19 transmission in (sub) tropical cities of Brazil. Science of the Total Environment. 2020;729:138862.
44. Jahangiri M, Jahangiri M, Najafgholipour M. The sensitivity and specificity analyses of ambient temperature and population size on the transmission rate of the novel coronavirus (COVID-19) in different provinces of Iran. Science of the Total Environment. 2020;728:138872.
45. Demongeot J, Flet-Berliac Y, Seligmann H. Temperature decreases spread parameters of the new Covid-19 case dynamics. Biology. 2020;9(5):94.
46. Wu Y, Jing W, Liu J, Ma Q, Yuan J, Wang Y, Du M, Liu M. Effects of temperature and humidity on the daily new cases and new deaths of COVID-19 in 166 countries. Science of the Total Environment. 2020;729:139051.
47. Rashed EA, Kodera S, Gomez-Tames J, Hirata A. Influence of absolute humidity, temperature and population density on COVID-19 spread and decay durations: Multi-prefecture study in Japan. International Journal of Environmental Research and Public Health. 2020;17(5354):1-14.
48. Lin J, Huang W, Wen M, Li D, Ma S, Hua J, Hu H, Yin S, Qian Y, Chen P, Zhang Q. Containing the spread of coronavirus disease 2019 (COVID-19): Meteorological factors and control strategies. Science of the Total Environment. 2020;744:140935.
49. Wang J, Tang K, Feng K, Lv W. High temperature and high humidity reduce the transmission of COVID-19; 2020.
Available:https://arxiv.org/ftp/arxiv/papers/2003/2003.05003.pdf
50. Jamil T, Alam I, Gojobori T, Duarte CM. No evidence for temperature-dependence of the COVID-19 epidemic. Frontiers in Public Health. 2020;8:436.
DOI: 10.3389/fpubh.2020.00436
Available:https://www.frontiersin.org/articles/10.3389/fpubh.2020.00436/full?&utm_source=Email_to_authors_&utm_medium=Email&utm_content=T1_11.5e1_author&utm_campaign=Email_publication&field=&journalName=Frontiers_in_Public_Health&id=546193
51. Abdollahi A, Rahbaralam M. Effect of temperature on the transmission of COVID-19: A machine learning case study in Spain. medRxiv; 2020.
Available:https://www.medrxiv.org/content/medrxiv/early/2020/05/06/2020.05.01.20087759.full.pdf
52. Briz-Redón Á, Serrano-Aroca Á. The effect of climate on the spread of the COVID-19 pandemic: A review of findings and statistical and modelling techniques. Progress in Physical Geography: Earth and Environment. 2020; 0309133320946302.

53. Domingos P. The role of Occam's razor in knowledge discovery. Data Mining and Knowledge Discovery. 1999;3(4):409-25.
54. Rasmussen CE, Ghahramani Z. Occam's razor. In Advances in Neural Information Processing Systems. 2001;294-300.
55. Rushdi AM. Occam's razor. KAU Engineering Magazine. 2011;5(1):58-61.
56. Rushdi AM, Al-Qwasmi MA. Exposition and comparison of two kinds of a posteriori analysis of fault trees. Journal of King Abdulaziz University: Computing and Information Technology. 2016;5(1-2):55-74.
57. Sonin AA. The physical basis of dimensional analysis. Cambridge MA; 2001.
Available:http://web.mit.edu/2.25/www/pdf/DA_unified.pdf
58. Banerjee S, Guedj J, Ribeiro RM, Moses M, Perelson AS. Estimating biologically relevant parameters under uncertainty for experimental within-host murine West Nile virus infection. Journal of the Royal Society Interface. 2016;13(20160130):1-13.
59. Punter MT, Hernandez-Garcia A, Kraft DJ, de Vries R, van der Schoot P. Self-assembly dynamics of linear virus-like particles: Theory and experiment. The Journal of Physical Chemistry B. 2016;120(26):6286-6297.
60. Zhou G, Xie ZW, Xu XH, Li Q. A new model of overall heat transfer coefficient of hot wax oil pipeline based on dimensionless experimental analysis. Case Studies in Thermal Engineering. 2020;100647:1-11.
(Available online April, 28, 2020)
61. Nittala A, Ghosh S, Stefanovski D, Bergman R, Wang X. Dimensional analysis of MINMOD leads to definition of the disposition index of glucose regulation and improved simulation algorithm. Biomedical Engineering Online. 2006;5(44):1-15.
62. Wonham MJ, de-Camino-Beck T, Lewis MA. An epidemiological model for West Nile virus: Invasion analysis and control applications. Proceedings of the Royal Society of London. Series B: Biological Sciences. 2004;271(1538):501-507.
63. Kilicman A. A fractional order SIR epidemic model for dengue transmission. Chaos, Solitons & Fractals. 2018;114:55-62.
64. Hasan S, Al-Zoubi A, Freihet A, Al-Smadi M, Momani S. Solution of fractional SIR epidemic model using residual power series method. Applied Mathematics and Information Sciences. 2019;13(2):153-61.
65. Kabir KA, Kuga K, Tanimoto J. Analysis of SIR epidemic model with information spreading of awareness. Chaos, Solitons & Fractals. 2019;119:118-25.
66. Naik PA, Yavuz M, Zu J. The role of prostitution on HIV transmission with memory: A modeling approach. Alexandria Engineering Journal. 2020;59(4):2513-2531.
67. Lowen AC, Mubareka S, Steel J, Palese P. Influenza virus transmission is dependent on relative humidity and temperature. PLoS Pathog. 2007;3(10);e151:1470-1476.
68. Lambrechts L, Paaijmans KP, Fansiri T, Carrington LB, Kramer LD, Thomas MB, Scott TW. Impact of daily temperature fluctuations on dengue virus transmission by *Aedes aegypti*. Proceedings of the National Academy of Sciences. 2011;108(18):7460-7465.
69. Paintsil E, Binka M, Patel A, Lindenbach BD, Heimer R. Hepatitis C virus maintains infectivity for weeks after drying on inanimate surfaces at room temperature: implications for risks of transmission. The Journal of Infectious Diseases. 2014;209(8):1205-1211.
70. Fauci AS, Morens DM. Zika virus in the Americas—yet another arbovirus threat. New England Journal of Medicine. 2016;374(7):601-604.
71. Faria NR, Kraemer MU, Hill SC, De Jesus JG, Aguiar RS, Iani FC, Xavier J, Quick J, Du Plessis L, Dellicour S, Theze J. Genomic and epidemiological monitoring of yellow fever virus transmission potential. Science. 2018;361(6405):894-899.
72. Noeiaghdam S, Suleman M, Budak H. Solving a modified nonlinear epidemiological model of computer viruses by homotopy analysis method. Mathematical Sciences. 2018;12(3):211-22.
73. Noeiaghdam S. A novel technique to solve the modified epidemiological model of computer viruses. SeMA Journal. 2019;76(1):97-108.
74. Raza A, Arif MS, Rafiq M, Bibi M, Naveed M, Iqbal MU, Butt Z, Naseem HA, Abbasi JN. Numerical treatment for stochastic computer virus model. Computer Modeling in Engineering & Sciences. 2019;120(2): 445-65.

75. Rushdi RA, Rushdi AM. Karnaugh-map utility in medical studies: The case of fetal malnutrition. International Journal of Mathematical, Engineering and Management Sciences. 2018;3(3):220-244.
76. Yan D, Davis GE. The turtleback diagram for conditional probability. arXiv preprint arXiv:1808.06884; 2018. Available:https://arxiv.org/pdf/1808.06884.pdf
77. Rushdi AM, Talmees FA. An exposition of the eight basic measures in diagnostic testing using several pedagogical tools. Journal of Advances in Mathematics and Computer Science. 2018;26(3):1-17.
78. Rushdi RA, Rushdi AM, Talmees FA. Novel pedagogical methods for conditional-probability computations in medical disciplines. Journal of Advances in Medicine and Medical Research. 2018;25(10):1-15.
79. Rushdi RA, Rushdi AM. Common fallacies of probability in medical context: A simple mathematical exposition. Journal of Advances in Medicine and Medical Research. 2018;26(1):1-21.
80. Rushdi AMA, Talmees FA. Computations of the eight basic measures in diagnostic testing. Chapter 6 in Advances in Mathematics and Computer Science, Vol. 2, Book Publishers International, Hooghly, West Bengal, India; 2019.
81. Rushdi RAM, Rushdi AMA. Mathematics and examples for avoiding common probability fallacies in medical disciplines. Chapter 11 in Current Trends in Medicine and Medical Research, Vol. 1, Book Publishers International, Hooghly, West Bengal, India; 2019.

Appendix A. Scalar DA Solution

This appendix provides a scalar formulation and interpretation for our present DA matrix solution, as a help to understanding and appreciating the all-matrix treatment via the modern approach given in the main text. First, we express each dimensionless product of the set of eight quantities in Table 3 in the form

$$\pi = k \ \ H^a \ {P_r}^b \ {c_a}^c \ \theta^d \ {V_P}^e \ {C_e}^f \ \ {E_{fs}}^g \ {I_P}^h, \tag{A1}$$

where k is a dimensionless constant, while a, b, c, d, e, f, g, and h are exponents yet to be partially determined or inter-related. If we denote by $[x]$ the dimension of the quantity x, we note that $[\pi] = [k] = 1$, where

$$[\pi] = \ (C \ \ L^{-6}\, T^3)^a \ (L)^b \ (L^3\, T^{-1})^c \ (C \ \ L^{-1}\, T)^d \ (L\, T^{-1})^e \ \ (T)^f \ \ \ (L^{-2})^g \ \ (1)^h, \tag{A2}$$

or, equivalently

$$C^0\, L^0\, T^0 = C^{a+d} \ \ L^{-6a+b+3c-d+e-2g} \ \ T^{3a-c+d-e+f} \,. \tag{A3}$$

The product π is dimensionless if

$$a + d \ \ = 0, \tag{A4a}$$

$$-6a + b \ + 3c - d + \ e \ \ - 2g \ \ \ \ = 0, \tag{A4b}$$

$$3a \ \ -c + d - e + f \ \ \ = 0. \tag{A4c}$$

In Equations (A4) we have eight unknowns (exponents a, b, c, d, e, f, g, and h), but only three conditions (equations). It will be seen shortly that these equations are linearly independent, *i.e.*, the dimensional matrix derived from them has a full rank of three. Thus, we might determine only three dependent exponents out of the eight exponents, supposedly in terms of the remaining five independent exponents. There are $\binom{8}{3}$ = 56 ways for choosing 3 objects out of 8 (with neither order nor repetition). Following the choice we made in Table 4, we take a, b and f as dependent exponents (basis variables), leaving the role of independent exponents (regime variables) to c, d, e, g and h.

Table 4 lists Equations (A4) in a dimensional-matrix form and then applies Gauss-Jordan elimination by performing elementary row operations to produce a 3 by 3 unit matrix under the symbols a, b and f. In Table 4, the dimensional matrix is supposed to be multiplied by a vector of indices, and the result is equated to a zero vector, *i.e*., Equations (A4) are rewritten as a single matrix equation in (A5), with unknowns rearranged as in Table 4

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -6 & 1 & 0 & -1 & 1 & 3 & -2 & 0 \\ 3 & 0 & 1 & 1 & -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ f \\ c \\ d \\ e \\ g \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{A5}$$

Matrix equation (A5) does not only have an awkward and ugly appearance, but it is also inconvenient to read, as it necessitates a 90-degree rotation of the exponent vector to make it coincide with each row of the matrix. Table 4 does not write the vector of indices as a column vector to the right of the dimensional matrix as in Eq. (A5), but writes it as a row vector on top of it [28,30,31,34]. It also omits the equality sign in Eq. (A5) and adds the zero vector in the R.H.S. of Eq. (A5) as an extra vector for the dimensional matrix resulting in an *augmented matrix*. Now, elementary row operations (solution-preserving operations) are applied to the whole rows of the augmented matrix, and are depicted by the labels to the left of the matrix. We note that we have justified our pre-supposition that the dimensional matrix is of full rank (rank three). The justification stems from the observation that the Gauss-Jordan procedure was completed in Table 4 without creating an all-0 row. The final result of Table 4 rewrites equations (A4) as

$$a + d = 0, \tag{A6a}$$

$$b + 5d + e + 3c - 2g = 0, \tag{A6b}$$

$$f - 2d - e - c = 0. \tag{A6c}$$

and hence it expresses each of the unknown eight indices in terms of the last five of them (c, d, e, g and h), thereby leading to the final result of (7) in the main text.

Peer-review history:
The peer review history for this paper can be accessed here:
http://www.sdiarticle4.com/review-history/61778