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On Summing Formulas for Generalized Fibonacci and Gaussian Generalized Fibonacci Numbers

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, closed forms of the summation formulas for generalized Fibonacci and Gaussian generalized Fibonacci numbers are presented. Then, some previous results are recovered as particular cases of the present results. As special cases, we give summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers and Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell-Lucas, Gaussian Jacobsthal, Gaussian Jacobsthal-Lucas numbers.

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1 INTRODUCTION

In 1965, Horadam [1] defined a generalization of Fibonacci sequence, that is, he defined a second-order linear recurrence sequence $\{W_n(W_0, W_1; r, s)\}$, or simply $\{W_n\}$, as follows:

$$W_n = rW_{n-1} + sW_{n-2}; \quad W_0 = a, \ W_1 = b, \quad (n \geq 2) \tag{1.1}$$

where W_0, W_1 are arbitrary complex numbers and r, s are real numbers, see also Horadam [2], [3] and [4]. Now these generalized Fibonacci numbers $\{W_n(a, b; r, s)\}$ are also called Horadam numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

For some specific values of a, b, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1. A few special case of generalized Fibonacci sequences.

Name of sequence	Notation: $W_n(a, b; r, s)$	No in oeis.org: [5]
Fibonacci	$F_n = W_n(0, 1; 1, 1)$	A000045
Lucas	$L_n = W_n(2, 1; 1, 1)$	A000032
Pell	$P_n = W_n(0, 1; 2, 1)$	A000129
Pell-Lucas	$Q_n = W_n(2, 2; 2, 1)$	A002203
Jacobsthal	$J_n = W_n(0, 1; 1, 2)$	A001045
Jacobsthal-Lucas	$j_n = W_n(2, 1; 1, 2)$	A014551

A Gaussian integer z is a complex number whose real and imaginary parts are both integers, i.e., $z = a + ib$, $a, b \in \mathbb{Z}$. If we use together sequences of integers defined recursively and Gaussian type integers, we obtain a new sequences of complex numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell-Lucas and Gaussian Jacobsthal numbers; Gaussian Padovan and Gaussian Pell-Padovan numbers; Gaussian Tribonacci numbers.

Gaussian generalized Fibonacci (Horadam) numbers $\{GW_n\}_{n\geq 0} = \{GW_n(GW_0, GW_1; r, s)\}_{n\geq 0}$ are defined by

$$GW_n = rGW_{n-1} + sGW_{n-2} \tag{1.2}$$

with the initial conditions

$$GW_0 = W_0 + (-\frac{r}{s}GW_0 + \frac{1}{s}GW_1)i, \ GW_1 = W_1 + W_0 i$$

not all being zero. The sequences $\{GW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GW_{-n} = -\frac{r}{s}GW_{-(n-1)} + \frac{1}{s}GW_{-(n-2)} = -\frac{r}{s}GW_{-n+1} + \frac{1}{s}GW_{-n+2}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.2) holds for all integer n. Note that for $n \geq 0$

$$GW_n = W_n + iW_{n-1}$$

and

$$GW_{-n} = W_{-n} + iW_{-n-1}.$$

For some specific values of W_0, W_1, r and s, it is worth presenting these special Gaussian Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 2) are used for the special cases of r, s and initial values.

Table 2. A few special case of generalized Gaussian Fibonacci sequences.

Name of sequence	Notation: $GW_n(GW_0, GW_1; r, s)$
Gaussian Fibonacci	$GF_n = GW_n(i, 1; 1, 1)$
Gaussian Lucas	$GL_n = GW_n(2 - i, 1 + 2i; 1, 1)$
Gaussian Pell	$GP_n = GW_n(i, 1; 2, 1)$
Gaussian Pell-Lucas	$GQ_n = GW_n(2 - 2i, 2 + 2i; 2, 1)$
Gaussian Jacobsthal	$GJ_n = GW_n(\frac{1}{2}i, 1; 1, 2)$
Gaussian Jacobsthal-Lucas	$Gj_n = GW_n(2 - \frac{1}{2}i, 1 + 2i; 1, 2)$

In this work, we investigate summation formulas of generalized Fibonacci and Gaussian generalized Fibonacci numbers. Some summing formulas of the Pell and Pell-Lucas numbers are well known and given in [6], [7], see also [8]. For linear sums of Fibonacci, Tribonacci, Tetranacci, Pentanacci and Hexanacci numbers, see [9], [10], [11], [12],[13], and [14] respectively.

2 SUMMING FORMULAS OF GENERALIZED FIBONACCI NUMBERS WITH POSITIVE SUBSCRIPTS

The following Theorem presents some linear summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 2.1. *For $n \geq 0$ we have the following formulas:*

(a) *(Sum of the generalized Fibonacci numbers) If $r + s - 1 \neq 0$, then*

$$\sum_{k=0}^{n} W_k = \frac{W_{n+2} + (1-r)W_{n+1} - W_1 + (r-1)W_0}{r + s - 1}.$$

(b) *If $(r - s + 1)(r + s - 1) \neq 0$ then*

$$\sum_{k=0}^{n} W_{2k} = \frac{(1-s)W_{2n+2} + rsW_{2n+1} + (s-1)W_2 - rsW_1 + (r^2 - s^2 + 2s - 1)W_0}{(r - s + 1)(r + s - 1)}$$

and

$$\sum_{k=0}^{n} W_{2k+1} = \frac{rW_{2n+2} + (s - s^2)W_{2n+1} - rW_2 + (r^2 + s - 1)W_1}{(r - s + 1)(r + s - 1)}.$$

(c) *If $r \neq 0 \wedge s = 1$ then*

$$\sum_{k=0}^{n} W_{2k} = \frac{W_{2n+1} - W_1 + rW_0}{r}$$

and

$$\sum_{k=0}^{n} W_{2k+1} = \frac{W_{2n+2} - W_2 + rW_1}{r}.$$

Note that (c) is a special case of (b).

Proof.

(a) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2}$$

i.e.

$$sW_{n-2} = W_n - rW_{n-1}$$

we obtain

$$\begin{array}{rcl} sW_0 & = & W_2 - rW_1 \\ sW_1 & = & W_3 - rW_2 \\ sW_2 & = & W_4 - rW_3 \\ & \vdots & \\ sW_{n-2} & = & W_n - rW_{n-1} \\ sW_{n-1} & = & W_{n+1} - rW_n \\ sW_n & = & W_{n+2} - rW_{n+1}. \end{array}$$

If we add the above equations by side by, we get

$$\sum_{k=0}^{n} W_k = \frac{W_{n+2} + (1-r)W_{n+1} - W_1 + (r-1)W_0}{r+s-1}.$$

(b) and (c) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2}$$

we obtain

$$\begin{array}{rcl} rW_3 & = & W_4 - sW_2 \\ rW_5 & = & W_6 - sW_4 \\ rW_7 & = & W_8 - sW_6 \\ & \vdots & \\ rW_{2n+1} & = & W_{2n+2} - sW_{2n}. \\ rW_{2n+3} & = & W_{2n+4} - sW_{2n+2} \end{array}$$

Now, if we add the above equations by side by, we get

$$r(-W_1 + \sum_{k=0}^{n} W_{2k+1}) = (W_{2n+2} - W_2 - W_0 + \sum_{k=0}^{n} W_{2k}) - s(-W_0 + \sum_{k=0}^{n} W_{2k})). \quad (2.1)$$

Similarly, using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2}$$

we write the following obvious equations;

$$\begin{array}{rcl} rW_2 & = & W_3 - sW_1 \\ rW_4 & = & W_5 - sW_3 \\ rW_6 & = & W_7 - sW_5 \\ rW_8 & = & W_9 - sW_7 \\ & \vdots & \\ rW_{2n} & = & W_{2n+1} - sW_{2n-1} \\ rW_{2n+2} & = & W_{2n+3} - sW_{2n+1}. \end{array}$$

Now, if we add the above equations by side by, we obtain

$$r(-W_0+\sum_{k=0}^{n}W_{2k})=(-W_1+\sum_{k=0}^{n}W_{2k+1})-s(-W_{2n+1}+\sum_{k=0}^{n}W_{2k+1}). \tag{2.2}$$

Then, solving the system (2.1)-(2.2), the required results of (b) and (c) follow.

Taking $r=s=1$ in Theorem 2.1 (a) and (b), we obtain the following Proposition.

Proposition 2.1. *If $r=s=1$ then for $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}W_k=W_{n+2}-W_1.$

(b) $\sum_{k=0}^{n}W_{2k}=W_{2n+1}-W_1+W_0.$

(c) $\sum_{k=0}^{n}W_{2k+1}=W_{2n+2}-W_2+W_1.$

From the above Proposition, we have the following Corollary which gives linear sum formulas of Fibonacci numbers (take $W_n=F_n$ with $F_0=0,F_1=1$).

Corollary 2.2. *For $n\geq 0,$ Fibonacci numbers have the following properties:*

(a) $\sum_{k=0}^{n}F_k=F_{n+2}-1.$

(b) $\sum_{k=0}^{n}F_{2k}=F_{2n+1}-1.$

(c) $\sum_{k=0}^{n}F_{2k+1}=F_{2n+2}.$

Taking $W_n=L_n$ with $L_0=2,L_1=1$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Lucas numbers.

Corollary 2.3. *For $n\geq 0,$ Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}L_k=\ L_{n+2}-1.$

(b) $\sum_{k=0}^{n}L_{2k}=L_{2n+1}+1.$

(c) $\sum_{k=0}^{n}L_{2k+1}\ L_{2n+2}-2.$

Taking $r=2,s=1$ in Theorem 2.1 (a) and (b), we obtain the following Proposition.

Proposition 2.2. *If $r=2,s=t=1$ then for $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}W_k=\frac{1}{2}(W_{n+2}-W_{n+1}-W_1+W_0).$

(b) $\sum_{k=0}^{n}W_{2k}=\ \frac{1}{2}(W_{2n+1}-W_1+2W_0).$

(c) $\sum_{k=0}^{n}W_{2k+1}=\ \frac{1}{2}(W_{2n+2}-W_2+2W_1).$

From the last Proposition, we have the following Corollary which gives linear sum formulas of Pell numbers (take $W_n=P_n$ with $P_0=0,P_1=1$).

Corollary 2.4. *For $n\geq 0,$ Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n}P_k=\frac{1}{2}(P_{n+2}-P_{n+1}-1).$

(b) $\sum_{k=0}^{n}P_{2k}=\frac{1}{2}(P_{2n+1}-1).$

(c) $\sum_{k=0}^{n}P_{2k+1}=\frac{1}{2}P_{2n+2}.$

Taking $W_n=Q_n$ with $Q_0=2,Q_1=2$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Pell-Lucas numbers.

Corollary 2.5. *For $n\geq 0,$ Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}Q_k=\frac{1}{2}(Q_{n+2}-Q_{n+1}).$

(b) $\sum_{k=0}^{n} Q_{2k} = \frac{1}{2}(Q_{2n+1} + 2)$.

(c) $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{2}(Q_{2n+2} - 2)$.

If $r = 1, s = 2$ then $(r - s + 1)(r + s - 1) = 0$ so we can't use Theorem 2.1 (b). In other words, the method of the proof Theorem 2.1 (b) can't be used to find $\sum_{k=0}^{n} W_{2k}$ and $\sum_{k=0}^{n} W_{2k+1}$. Therefore we need another method to find them which is given in the following Theorem.

Theorem 2.6. *If $r = 1, s = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = \frac{1}{2}(W_{n+2} - W_1)$.

(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{3}(2W_{2n+2} - 2W_{2n+1} - W_0 + (-W_1 + 2W_0)n)$.

(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{6}(-W_{2n+2} + 10W_{2n+1} - 3W_1 + 2W_0 + (2W_1 - 4W_0)\, n)$.

Proof.

(a) Taking $r = 1, s = 2$ in Theorem 2.1 (a) we obtain (a).

(b) and (c) (b) and (c) can be proved by mathematical induction.

From the last Theorem we have the following Corollary which gives linear sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 2.7. *For $n \geq 0$, Jacobsthal numbers have the following property:*

(a) $\sum_{k=0}^{n} J_k = \frac{1}{2}(J_{n+2} - 1)$.

(b) $\sum_{k=0}^{n} J_{2k} = \frac{1}{3}(2J_{2n+2} - 2J_{2n+1} - n)$.

(c) $\sum_{k=0}^{n} J_{2k+1} = \frac{1}{6}(-J_{2n+2} + 10J_{2n+1} - 3 + 2n)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last Theorem, we have the following Corollary which presents linear sum formulas of Jacobsthal-Lucas numbers.

Corollary 2.8. *For $n \geq 0$, Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=0}^{n} j_k = \frac{1}{2}(j_{n+2} - 1)$.

(b) $\sum_{k=0}^{n} j_{2k} = \frac{1}{3}(2j_{2n+2} - 2j_{2n+1} - 2 + 3n)$.

(c) $\sum_{k=0}^{n} j_{2k+1} = \frac{1}{6}(-j_{2n+2} + 10j_{2n+1} + 1 - 6n)$.

3 SUMMING FORMULAS OF GENERALIZED FIBONACCI NUMBERS WITH NEGATIVE SUBSCRIPTS

The following Theorem presents some linear summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 3.1. *For $n \geq 1$ we have the following formulas:*

(a) *(Sum of the generalized Fibonacci numbers with negative indices) If $r + s - 1 \neq 0$, then*

$$\sum_{k=1}^{n} W_{-k} = \frac{-(r + s)W_{-n-1} - sW_{-n-2} + W_1 + (1 - r)W_0}{r + s - 1}.$$

(b) *If* $(r-s+1)(r+s-1)\neq 0$ *then*

$$\sum_{k=1}^{n}W_{-2k}=\frac{(s-1)W_{-2n}-rsW_{-2n-1}+rW_{1}+(1-s-r^{2})W_{0}}{(r-s+1)(r+s-1)}$$

and

$$\sum_{k=1}^{n}W_{-2k+1}=\frac{-rW_{-2n}+(s^{2}-s)W_{-2n-1}+(1-s)W_{1}+rsW_{0}}{(r-s+1)(r+s-1)}.$$

(c) *If* $r\neq 0\wedge s=1$ *then*

$$\sum_{k=1}^{n}W_{-2k}=\frac{1}{r}(-W_{-2n-1}+W_{1}-rW_{0})$$

and

$$\sum_{k=1}^{n}W_{-2k+1}=\frac{1}{r}(-W_{-2n}+W_{0}).$$

Note that (c) is a special case of (b).

Proof.

(a) Using the recurrence relation

$$W_{-n+2}=rW_{-n+1}+sW_{-n}$$

i.e.

$$sW_{-n}=W_{-n+2}-rW_{-n+1}$$

or

$$W_{-n}=\frac{1}{s}W_{-n+2}-\frac{r}{s}W_{-n+1}$$

we obtain

$$\begin{array}{rcl}
sW_{-n} & = & W_{-n+2}-rW_{-n+1}\\
sW_{-n+1} & = & W_{-n+3}-rW_{-n+2}\\
sW_{-n+2} & = & W_{-n+4}-rW_{-n+3}\\
 & \vdots & \\
sW_{-3} & = & W_{-1}-rW_{-2}\\
sW_{-2} & = & W_{0}-rW_{-1}\\
sW_{-1} & = & W_{1}-rW_{0}.
\end{array}$$

If we add the above equations by side by, we get

$$\sum_{k=1}^{n}W_{-k}=\frac{(1-r-s)W_{-n-1}-(r+s)W_{-n-2}-sW_{-n-3}+W_{1}+(1-r)W_{0}}{r+s-1}.$$

(b) and (c) Using the recurrence relation

$$W_{-n+2}=rW_{-n+1}+sW_{-n}$$

i.e.

$$rW_{-n+1}=W_{-n+2}-sW_{-n}$$

we obtain

$$\begin{array}{rcl} rW_{-2n+1} & = & W_{-2n+2} - sW_{-2n} \\ rW_{-2n+3} & = & W_{-2n+4} - sW_{-2n+2} \\ rW_{-2n+5} & = & W_{-2n+6} - sW_{-2n+4} \\ & \vdots & \\ rW_{-5} & = & W_{-4} - sW_{-6} \\ rW_{-3} & = & W_{-2} - sW_{-4} \\ rW_{-1} & = & W_0 - sW_{-2}. \end{array}$$

If we add the above equations by side by, we get

$$r\sum_{k=1}^{n} W_{-2k+1} = (-W_{-2n} + W_0 + \sum_{k=1}^{n} W_{-2k}) - s(\sum_{k=1}^{n} W_{-2k}). \tag{3.1}$$

Similarly, using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n}$$

i.e.

$$rW_{-2n} = W_{-2n+1} - sW_{-2n-1}$$

we obtain

$$\begin{array}{rcl} rW_{-2n} & = & W_{-2n+1} - sW_{-2n-1} \\ rW_{-2n+2} & = & W_{-2n+3} - sW_{-2n+1} \\ rW_{-2n+4} & = & W_{-2n+5} - sW_{-2n+3} \\ & \vdots & \\ rW_{-4} & = & W_{-3} - sW_{-5} \\ rW_{-2} & = & W_{-1} - sW_{-3}. \end{array}$$

If we add the above equations by side by, we get

$$r\sum_{k=1}^{n} W_{-2k} = (\sum_{k=1}^{n} W_{-2k+1}) - s(W_{-2n-1} - W_{-1} + \sum_{k=1}^{n} W_{-2k+1}).$$

Since

$$W_{-1} = (-\frac{r}{s} \times W_0 + \frac{1}{s} W_1)$$

it follows that

$$r\sum_{k=1}^{n} W_{-2k} = (\sum_{k=1}^{n} W_{-2k+1}) - s(W_{-2n-1} - (-\frac{r}{s} \times W_0 + \frac{1}{s} W_1) + \sum_{k=1}^{n} W_{-2k+1}). \tag{3.2}$$

Then, solving system (3.1)-(3.2) the required result of (b) and (c) follow.

Taking $r = s = 1$ in Theorem 3.1 (a) and (b), we obtain the following Proposition.

Proposition 3.1. *If $r = s = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k} = -2W_{-n-1} - W_{-n-2} + W_1$.

(b) $\sum_{k=1}^{n} W_{-2k} = -W_{-2n-1} + W_1 - W_0$.

(c) $\sum_{k=1}^{n} W_{-2k+1} = -W_{-2n} + W_0.$

From the above Proposition, we have the following Corollary which gives linear sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 3.2. *For $n \geq 1$, Fibonacci numbers have the following properties.*

(a) $\sum_{k=1}^{n} F_{-k} = -2F_{-n-1} - F_{-n-2} + 1.$

(b) $\sum_{k=1}^{n} F_{-2k} = -F_{-2n-1} + 1.$

(c) $\sum_{k=1}^{n} F_{-2k+1} = -F_{-2n}.$

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Lucas numbers.

Corollary 3.3. *For $n \geq 1$, Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} L_{-k} = -2L_{-n-1} - L_{-n-2} + 1.$

(b) $\sum_{k=1}^{n} L_{-2k} = -L_{-2n-1} - 1.$

(c) $\sum_{k=1}^{n} L_{-2k+1} = -L_{-2n} + 2.$

Taking $r = 2, s = 1$ in Theorem 3.1 (a) and (b), we obtain the following Proposition.

Proposition 3.2. *If $r = 2, s = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k} = \frac{1}{2}(-3W_{-n-1} - W_{-n-2} + W_1 - W_0).$

(b) $\sum_{k=1}^{n} W_{-2k} = \frac{1}{2}(-W_{-2n-1} + W_1 - 2W_0).$

(c) $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{2}(-W_{-2n} + W_0).$

From the last Proposition, we have the following Corollary which gives linear sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 3.4. *For $n \geq 1$, Pell numbers have the following properties.*

(a) $\sum_{k=1}^{n} P_{-k} = \frac{1}{2}(-3P_{-n-1} - P_{-n-2} + 1).$

(b) $\sum_{k=1}^{n} P_{-2k} = \frac{1}{2}(-P_{-2n-1} + 1).$

(c) $\sum_{k=1}^{n} P_{-2k+1} = -\frac{1}{2}P_{-2n}.$

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Pell-Lucas numbers.

Corollary 3.5. *For $n \geq 1$, Pell-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} Q_{-k} = \frac{1}{2}(-3Q_{-n-1} - Q_{-n-2}).$

(b) $\sum_{k=1}^{n} Q_{-2k} = \frac{1}{2}(-Q_{-2n-1} - 2).$

(c) $\sum_{k=1}^{n} Q_{-2k+1} = \frac{1}{2}(-Q_{-2n} + 2).$

If $r = 1, s = 2$ then $(r - s + 1)(r + s - 1) = 0$ so we can't use Theorem 3.1 (b). In other words, the method of the proof Theorem 3.1 (b) can't be used to find $\sum_{k=0}^{n} W_{2k}$ and $\sum_{k=0}^{n} W_{2k+1}$. Therefore we need another method to find them which is given in the following Theorem.

Theorem 3.6. *If $r = 0, s = 2, t = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k} = \frac{1}{2}(-3W_{-n-1} - 2W_{-n-2} + W_1).$

(b) $\sum_{k=1}^{n} W_{-2k} = \frac{1}{512}(-213W_{-n-1} - 214W_{-n-2} + (73W_1 + 70W_0) + (-174W_1 + 340W_0)n).$

(c) $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{256}(-213W_{-n-1} - 214W_{-n-2} + (73W_1 + 70W_0) + (-172W_0 + 82W_1)n).$

Proof.

(a) Taking $r = 1, s = 2$ in Theorem 3.1 (a) we obtain (a).

(b) and (c) (b) and (c) can be proved by mathematical induction.

From the last Theorem, we have the following Corollary which gives sum formula of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 3.7. *For $n \geq 1$, Jacobsthal numbers have the following property:*

(a) $\sum_{k=1}^{n} J_{-k} = \frac{1}{2}(-3J_{-n-1} - 2J_{-n-2} + J_1).$

(b) $\sum_{k=1}^{n} J_{-2k} = \frac{1}{512}(-213J_{-n-1} - 214J_{-n-2} + 73 - 174n).$

(c) $\sum_{k=1}^{n} J_{-2k+1} = \frac{1}{256}(-213J_{-n-1} - 214J_{-n-2} + 73 + 82n).$

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last Proposition, we have the following Corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 3.8. *For $n \geq 1$, Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=1}^{n} j_{-k} = \frac{1}{2}(-3j_{-n-1} - 2j_{-n-2} + 1).$

(b) $\sum_{k=1}^{n} j_{-2k} = \frac{1}{512}(-213j_{-n-1} - 214j_{-n-2} + 213 + 506n).$

(c) $\sum_{k=1}^{n} j_{-2k+1} = \frac{1}{256}(-213j_{-n-1} - 214j_{-n-2} + 213 - 262n).$

4 SUMMING FORMULAS OF GAUSSIAN GENERALIZED FIBONACCI NUMBERS WITH POSITIVE SUBSCRIPTS

The following Theorem presents some linear summing formulas of Gaussian generalized Fibonacci numbers with positive subscripts.

Theorem 4.1. *For $n \geq 0$ we have the following formulas:*

(a) *(Sum of the generalized Gaussian Fibonacci numbers) If $r + s - 1 \neq 0$, then*

$$\sum_{k=0}^{n} GW_k = \frac{GW_{n+2} + (1-r)GW_{n+1} - GW_1 + (r-1)GW_0}{r+s-1}.$$

(b) *If $(r - s + 1)(r + s - 1) \neq 0$ then*

$$\sum_{k=0}^{n} GW_{2k} = \frac{(1-s)GW_{2n+2} + rsGW_{2n+1} + (s-1)GW_2 - rsGW_1 + (r^2 - s^2 + 2s - 1)GW_0}{(r-s+1)(r+s-1)}$$

and

$$\sum_{k=0}^{n} GW_{2k+1} = \frac{rGW_{2n+2} + (s - s^2)GW_{2n+1} - rGW_2 + (r^2 + s - 1)GW_1}{(r-s+1)(r+s-1)}.$$

(c) *If $r \neq 0 \wedge s = 1$ then*

$$\sum_{k=0}^{n} GW_{2k} = \frac{GW_{2n+1} - GW_1 + rGW_0}{r}$$

and

$$\sum_{k=0}^{n} GW_{2k+1} = \frac{GW_{2n+2} - GW_2 + rGW_1}{r}.$$

Note that (c) is a special case of (b).

Proof. The proof can be given exactly as in the proof of Theorem 2.1.
Taking $r = s = 1$ in Theorem 4.1 (a) and (b), we obtain the following Proposition.

Proposition 4.1. *If $r = s = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} GW_k = GW_{n+2} - GW_1.$

(b) $\sum_{k=0}^{n} GW_{2k} = GW_{2n+1} - GW_1 + GW_0.$

(c) $\sum_{k=0}^{n} GW_{2k+1} = GW_{2n+2} - GW_2 + GW_1.$

From the above Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Fibonacci numbers (take $GW_n = GF_n$ with $GF_0 = i, GF_1 = 1$).

Corollary 4.2. *For $n \geq 0$, Gaussian Fibonacci numbers have the following properties:*

(a) $\sum_{k=0}^{n} GF_k = GF_{n+2} - 1.$

(b) $\sum_{k=0}^{n} GF_{2k} = GF_{2n+1} - (1 - i\ ).$

(c) $\sum_{k=0}^{n} GF_{2k+1} = \ GF_{2n+2} - i.$

Taking $GW_n = GL_n$ with $GL_0 = 2 - i, GL_1 = 1 + 2i$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Gaussian Lucas numbers.

Corollary 4.3. *For $n \geq 0$, Gaussian Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} GL_k = GL_{n+2} - (1 + 2i).$

(b) $\sum_{k=0}^{n} GL_{2k} = GL_{2n+1} + (1 - 3i).$

(c) $\sum_{k=0}^{n} GL_{2k+1} = GL_{2n+2} + (-2 + i).$

Taking $r = 2, s = 1$ in Theorem 4.1 (a) and (b), we obtain the following Proposition.

Proposition 4.2. *If $r = 2, s = t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} GW_k = \frac{1}{2}(GW_{n+2} - GW_{n+1} - GW_1 + GW_0).$

(b) $\sum_{k=0}^{n} GW_{2k} = \frac{1}{2}(GW_{2n+1} - GW_1 + 2GW_0).$

(c) $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{2}(GW_{2n+2} - GW_2 + 2GW_1).$

From the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Pell numbers (take $GW_n = GP_n$ with $GP_0 = i, GP_1 = 1$).

Corollary 4.4. *For $n \geq 0$, Gaussian Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} GP_k = \frac{1}{2}(GP_{n+2} - GP_{n+1} - (1 - i)).$

(b) $\sum_{k=0}^{n} GP_{2k} = \frac{1}{2}(GP_{2n+1} - (1 - 2i)).$

(c) $\sum_{k=0}^{n} GP_{2k+1} = \frac{1}{2}(GP_{2n+2} - i).$

Taking $GW_n = GQ_n$ with $GQ_0 = 2 - 2i, GQ_1 = 2 + 2i$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Gaussian Pell-Lucas numbers.

Corollary 4.5. *For $n \geq 0$, Gaussian Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} GQ_k = \frac{1}{2}(GQ_{n+2} - GQ_{n+1} - 4i).$

(b) $\sum_{k=0}^{n} GQ_{2k} = \frac{1}{2}(GQ_{2n+1} + (2 - 6i)).$

(c) $\sum_{k=0}^{n} GQ_{2k+1} = \frac{1}{2}(GQ_{2n+2} - (2 - 2i)).$

If $r=1,s=2$ then $(r-s+1)(r+s-1)=0$ so we can't use Theorem 4.1 (b). In other words, the method of the proof Theorem 4.1 (b) can't be used to find $\sum_{k=0}^{n}GW_{2k}$ and $\sum_{k=0}^{n}GW_{2k+1}$. Therefore we need another method to find them which is given in the following Theorem.

Theorem 4.6. *If $r=1,s=2$ then for $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}GW_{k}=\frac{1}{2}(GW_{n+2}-GW_{1})$.

(b) $\sum_{k=0}^{n}GW_{2k}=\frac{1}{3}(2GW_{2n+2}-2GW_{2n+1}-GW_{0}+(-GW_{1}+2GW_{0})n)$.

(c) $\sum_{k=0}^{n}GW_{2k+1}=\frac{1}{6}(-GW_{2n+2}+10GW_{2n+1}-3GW_{1}+2GW_{0}+(2GW_{1}-4GW_{0})\,n)$.

Proof.

(a) Taking $r=1,s=2$ in Theorem 4.1 (a) we obtain (a).

(b) and (c) (b) and (c) can be proved by mathematical induction.

From the last Theorem we have the following Corollary which gives linear sum formulas of Gaussian Jacobsthal numbers (take $GW_n=GJ_n$ with $GJ_0=\frac{1}{2}i,GJ_1=1$).

Corollary 4.7. *For $n\geq 0$, Gaussian Jacobsthal numbers have the following property:*

(a) $\sum_{k=0}^{n}GJ_{k}=\frac{1}{2}(GJ_{n+2}-1)$.

(b) $\sum_{k=0}^{n}GJ_{2k}=\frac{1}{3}(2GJ_{2n+2}-2GJ_{2n+1}-\frac{1}{2}i-(1-i)n)$.

(c) $\sum_{k=0}^{n}GJ_{2k+1}=\frac{1}{6}(-GJ_{2n+2}+10GJ_{2n+1}+(-3+i)+(2-2i)n)$.

Taking $GW_n=Gj_n$ with $Gj_0=2-\frac{1}{2}i,Gj_1=1+2i$ in the last Theorem, we have the following Corollary which presents linear sum formulas of Gaussian Jacobsthal-Lucas numbers.

Corollary 4.8. *For $n\geq 0$, Gaussian Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=0}^{n}Gj_{k}=\frac{1}{2}(Gj_{n+2}-(1+2i))$.

(b) $\sum_{k=0}^{n}Gj_{2k}=\frac{1}{3}(2Gj_{2n+2}-2Gj_{2n+1}-(2-\frac{1}{2}i)+(3-3i)n)$.

(c) $\sum_{k=0}^{n}Gj_{2k+1}=\frac{1}{6}(-Gj_{2n+2}+10Gj_{2n+1}+(1-7i)+(-6+6i)n)$.

5 SUMMING FORMULAS OF GAUSSIAN GENERALIZED FIBONACCI NUMBERS WITH NEGATIVE SUBSCRIPTS

The following Theorem presents some linear summing formulas of Gaussian generalized Fibonacci numbers with negative subscripts.

Theorem 5.1. *For $n\geq 1$ we have the following formulas:*

(a) *(Sum of the generalized Gaussian Fibonacci numbers with negative indices) If $r+s-1\neq 0$, then*

$$\sum_{k=1}^{n}GW_{-k}=\frac{-(r+s)GW_{-n-1}-sGW_{-n-2}+GW_{1}+(1-r)GW_{0}}{r+s-1}.$$

(b) *If $(r-s+1)(r+s-1)\neq 0$ then*

$$\sum_{k=1}^{n}GW_{-2k}=\frac{(s-1)GW_{-2n}-rsGW_{-2n-1}+rGW_{1}+(1-s-r^{2})GW_{0}}{(r-s+1)(r+s-1)}$$

and

$$\sum_{k=1}^{n}GW_{-2k+1}=\frac{-rGW_{-2n}+(s^{2}-s)GW_{-2n-1}+(1-s)GW_{1}+rsGW_{0}}{(r-s+1)(r+s-1)}.$$

(c) *If* $r \neq 0 \wedge s = 1$ *then*

$$\sum_{k=1}^{n} GW_{-2k} = \frac{1}{r}(-GW_{-2n-1} + GW_1 - rGW_0)$$

and

$$\sum_{k=1}^{n} GW_{-2k+1} = \frac{1}{r}(-GW_{-2n} + GW_0).$$

Note that (c) is a special case of (b).

Proof. The proof can be given exactly as in the proof of Theorem 3.1.
Taking $r = s = 1$ in Theorem 5.1 (a) and (b), we obtain the following Proposition.

Proposition 5.1. *If* $r = s = 1$ *then for* $n \geq 1$ *we have the following formulas:*

(a) $\sum_{k=1}^{n} GW_{-k} = -2GW_{-n-1} - GW_{-n-2} + GW_1.$

(b) $\sum_{k=1}^{n} GW_{-2k} = -GW_{-2n-1} + GW_1 - GW_0.$

(c) $\sum_{k=1}^{n} GW_{-2k+1} = -GW_{-2n} + GW_0.$

From the above Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Fibonacci numbers (take $GW_n = GF_n$ with $GF_0 = i, GF_1 = 1$).

Corollary 5.2. *For* $n \geq 1$, *Gaussian Fibonacci numbers have the following properties.*

(a) $\sum_{k=1}^{n} GF_{-k} = -2GF_{-n-1} - GF_{-n-2} + 1.$

(b) $\sum_{k=1}^{n} GF_{-2k} = -GF_{-2n-1} + 1 - i.$

(c) $\sum_{k=1}^{n} GF_{-2k+1} = -GF_{-2n} + i.$

Taking $GW_n = GL_n$ with $GL_0 = 2 - i, GL_1 = 1 + 2i$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Gaussian Lucas numbers.

Corollary 5.3. *For* $n \geq 1$, *Gaussian Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} GL_{-k} = -2GL_{-n-1} - GL_{-n-2} + (1 + 2i).$

(b) $\sum_{k=1}^{n} GL_{-2k} = -GL_{-2n-1} + (-1 + 3i).$

(c) $\sum_{k=1}^{n} GL_{-2k+1} = -GL_{-2n} + (2 - i).$

Taking $r = 2, s = 1$ in Theorem 5.1 (a) and (b), we obtain the following Proposition.

Proposition 5.2. *If* $r = 2, s = 1$ *then for* $n \geq 1$ *we have the following formulas:*

(a) $\sum_{k=1}^{n} GW_{-k} = \frac{1}{2}(-3GW_{-n-1} - GW_{-n-2} + GW_1 - GW_0).$

(b) $\sum_{k=1}^{n} GW_{-2k} = \frac{1}{2}(-GW_{-2n-1} + GW_1 - 2GW_0).$

(c) $\sum_{k=1}^{n} GW_{-2k+1} = \frac{1}{2}(-GW_{-2n} + GW_0).$

From the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Pell numbers (take $GW_n = GP_n$ with $GP_0 = i, GP_1 = 1$).

Corollary 5.4. *For* $n \geq 1$, *Gaussian Pell numbers have the following properties.*

(a) $\sum_{k=1}^{n} GP_{-k} = \frac{1}{2}(-3GP_{-n-1} - GP_{-n-2} + 1).$

(b) $\sum_{k=1}^{n} GP_{-2k} = \frac{1}{2}(-GP_{-2n-1} + 1).$

(c) $\sum_{k=1}^{n} GP_{-2k+1} = -\frac{1}{2}GP_{-2n}.$

Taking $GW_n = GQ_n$ with $GQ_0 = 2 - 2i, GQ_1 = 2 + 2i$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Gaussian Pell-Lucas numbers.

Corollary 5.5. *For $n \geq 1,$ Gaussian Pell-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} GQ_{-k} = \frac{1}{2}(-3GQ_{-n-1} - GQ_{-n-2} + 4i).$

(b) $\sum_{k=1}^{n} GQ_{-2k} = \frac{1}{2}(-GQ_{-2n-1} - (2 - 6i)).$

(c) $\sum_{k=1}^{n} GQ_{-2k+1} = \frac{1}{2}(-GQ_{-2n} + (2 - 2i)).$

If $r = 1, s = 2$ then $(r - s + 1)(r + s - 1) = 0$ so we can't use Theorem 5.1 (b). In other words, the method of the proof Theorem 5.1 (b) can't be used to find $\sum_{k=0}^{n} GW_{2k}$ and $\sum_{k=0}^{n} GW_{2k+1}$. Therefore we need another method to find them which is given in the following Theorem.

Theorem 5.6. *If $r = 0, s = 2, t = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} GW_{-k} = \frac{1}{2}(-3GW_{-n-1} - 2GW_{-n-2} + GW_1).$

(b) $\sum_{k=1}^{n} GW_{-2k} = \frac{1}{512}(-213GW_{-n-1} - 214GW_{-n-2} + (73GW_1 + 70GW_0) + (-174GW_1 + 340GW_0)n).$

(c) $\sum_{k=1}^{n} GW_{-2k+1} = \frac{1}{256}(-213GW_{-n-1} - 214GW_{-n-2} + (73GW_1 + 70GW_0) + (-172GW_0 + 82GW_1)n).$

Proof.

(a) Taking $r = 1, s = 2$ in Theorem 5.1 (a) we obtain (a).

(b) and (c) (b) and (c) can be proved by mathematical induction.

From the last Theorem, we have the following Corollary which gives sum formula of Gaussian Jacobsthal numbers (take $GW_n = GJ_n$ with $GJ_0 = \frac{1}{2}i, GJ_1 = 1$).

Corollary 5.7. *For $n \geq 1,$ Gaussian Jacobsthal numbers have the following property:*

(a) $\sum_{k=1}^{n} GJ_{-k} = \frac{1}{2}(-3GJ_{-n-1} - 2GJ_{-n-2} + 1).$

(b) $\sum_{k=1}^{n} GJ_{-2k} = \frac{1}{512}(-213GJ_{-n-1} - 214GJ_{-n-2} + (73 + 35i) + (-174 + 170i)n).$

(c) $\sum_{k=1}^{n} GJ_{-2k+1} = \frac{1}{256}(-213GJ_{-n-1} - 214GJ_{-n-2} + (73 + 35i) + (82 - 86i)n).$

Taking $GW_n = Gj_n$ with $Gj_0 = 2 - \frac{1}{2}i, Gj_1 = 1 + 2i$ in the last Proposition, we have the following Corollary which presents sum formulas of Gaussian Jacobsthal-Lucas numbers.

Corollary 5.8. *For $n \geq 1,$ Gaussian Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=1}^{n} Gj_{-k} = \frac{1}{2}(-3Gj_{-n-1} - 2Gj_{-n-2} + (1 + 2i)).$

(b) $\sum_{k=1}^{n} Gj_{-2k} = \frac{1}{512}(-213Gj_{-n-1} - 214Gj_{-n-2} + (213 + 111i) + (506 - 518i)n).$

(c) $\sum_{k=1}^{n} Gj_{-2k+1} = \frac{1}{256}(-213Gj_{-n-1} - 214Gj_{-n-2} + (213 + 111i) + (-262 + 250i)n).$

6 CONCLUSION

In this work, a number of linear and a few non-linear sum identities were discovered and proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written linear sum identities in terms of the generalized Fibonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the special cases of the generalized Fibonacci Gaussiann generalized Fibonacci sequences such as Fibonacci-Lucas sequence and Gaussian Fibonacci-Lucas sequence.

All the listed identities may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered. Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. See, for example, the articles [15], [16].

Our next publication will be about summation formulas for generalized Tribonacci and Gaussian generalized Tribonacci numbers using similar methods of this paper. Also we plan to investigate summation formulas for generalized Tetranacci, Gaussian generalized Tetranacci numbers and for generalized Pentanacci, Gaussian generalized Pentanacci numbers.

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COMPETING INTERESTS

Author has declared that no competing interests exist.

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